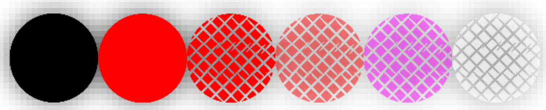
Illustrating Geometric Algebra and Differential Geometry in 5D Color Space

Geometric Algebra (GA) is popular for its immediate geometric interpretations of algebraic objects and operations. It is based on Clifford Algebra on vector spaces and extends linear algebra of vectors by operations such as an invertible product, i.e. divisions by vectors. This formalism allows for a complete algebra on vectors same as for scalar or complex numbers. It is particularly suitable for rotations in arbitrary dimensions. In Euclidean 3D space quaternions are known to be numerically superior to rotation matrices and already widely used in computer graphics. However, their meaning beyond its numerical formalism often remains mysterious. GA allows for an intuitive interpretation in terms of planes of rotations and extends this concept to arbitrary dimensions by embedding vectors into a higher dimensional, but still intuitively graspable space of multi-vectors. However, out intuition of more than three spatial dimensions is deficient. The space of colors forms a vector space as well, though one of non-spatial nature, but spun by the primary colors red, green, blue. The GA formalism can be applied here as well, amalgamating surprisingly with the notion of vectors and co-vectors known from differential geometry: tangential vectors on a manifold correspond to additive colors red/green/blue, whereas co-vectors from the co-tangential space correspond to subtractive primary colors magenta, yellow, cyan. GA in turn considers vectors, bi-vectors and anti-vectors as part of its generalized multi-vector zoo of algebraic objects. In 3D space vectors, anti-vectors, bi-vectors and co-vectors are all three-dimensional objects that can be identified with each other, so their distinction is concealed. Confusions arise from notions such as “normal vectors” vs. “axial vectors”. Higher dimensional spaces exhibit the differences more clearly. Using colors instead of spatial dimensions we can expand our intuition by considering "transparency" as an independent, four-dimensional property of a color vector. We can thereby explore 4D GA alternatively to spacetime in special/general relativity. However, even in 4D possibly confusing ambiguities remain between vectors, co-vectors, bi-vectors and bi-co-vectors: bi-vectors and bi-co-vectors - both six-dimensional objects - are visually equivalent. They become unequivocal only in five or higher dimensions. Envisioning five-dimensional geometry is even more challenging to the human mind, but in color space we can add another property, "texture" to constitute a five-dimensional vector space. The



properties of a bi-vector and a bi-co-vector becomes evident there: We can still study all possible combinations of colors/transparency/texture visually. This higher-dimensional yet intuitive approach demonstrates the need to distinguish among different kinds of vectors before identifying them in special situations, which also clarifies the meanings of algebraic objects in 3D Euclidean space and allows for better formulations of algorithms in 3D.

Werner Benger is technical director at AirborneHydroMapping GmbH (since 2014), an Austrian-based company developing software for processing and analyzing huge, high-resolution geoscientific data utilizing methods inspired by general relativity and geometric algebra. These design concepts evolved from prior work on visualizing numerical relativity such as colliding black holes calculated by high-performance supercomputer simulations. performed by the Max-Planck-Institute for Gravitational Physics in Potsdam and the Zuse-Institute Berlin (1997-2005; PhD from Free University Berlin, 2004). He has further refined those mathematical and computational concepts during practical work on scientific visualization of hurricanes, computational fluid dynamics and more at the Center for Computation & Technology at Louisiana State University (2005-2013).