

Radon and Mojette Projections' Equivalence for Tomographic Reconstruction using Linear Systems

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Context

Medical or industrial CT scanner :

- Measures of X-Rays attenuation through an object
- 1D projection of the object
- 2D slice reconstructed using Radon theorem

Radon Theorem [Radon, 1919]:

- Defined in continuous domain
- Images are in discrete domain \Rightarrow approximations

Two approaches :

- Improve reconstructed image quality
- Directly implement discrete reconstructions

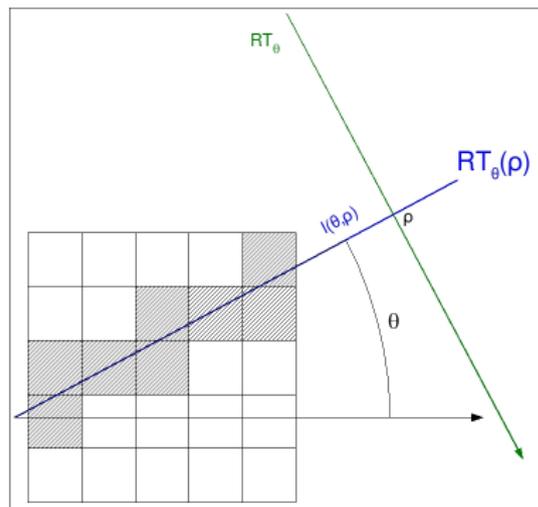
Plan

- Introduce the Radon theorem (discrete implementation and limitations)
- Present the Mojette transform (discrete transform, exact reconstruction, different from CT scan acquisition)
- Modelisation of the Radon acquisition to make it compatible with Mojette reconstruction

Discrete Radon Transform and Retroprojection

$$RT_{\theta}(\rho) = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} \mathcal{I}(k, l) \text{kernel}(\rho - k\cos\theta + l\sin\theta)$$

$$\mathcal{I}'(k, l) = \sum_{\theta=0}^{\pi} \sum_{\rho=-\infty}^{\infty} RT_{\theta}(\rho) \text{kernel}(\rho - k\cos\theta + l\sin\theta)$$



from Radon Transform to Sinogram

- $RT_{\theta}(\rho)$ is a projection value

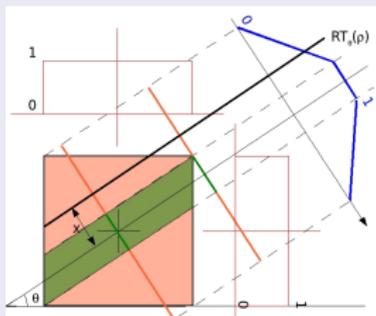
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Usual Kernels

- Dirac Impulse
- B-Spline 0 Kernel :



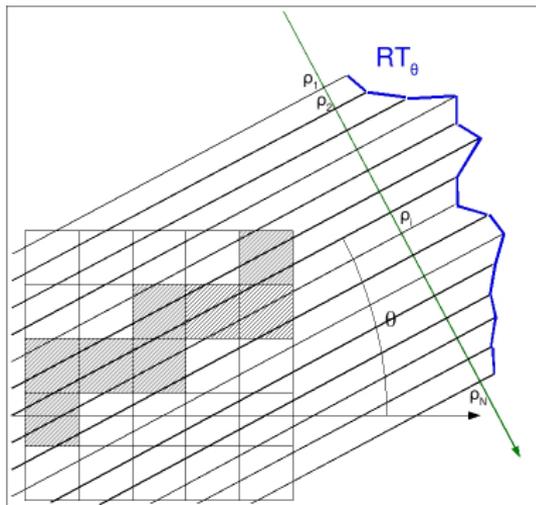
from Radon Transform to Sinogram

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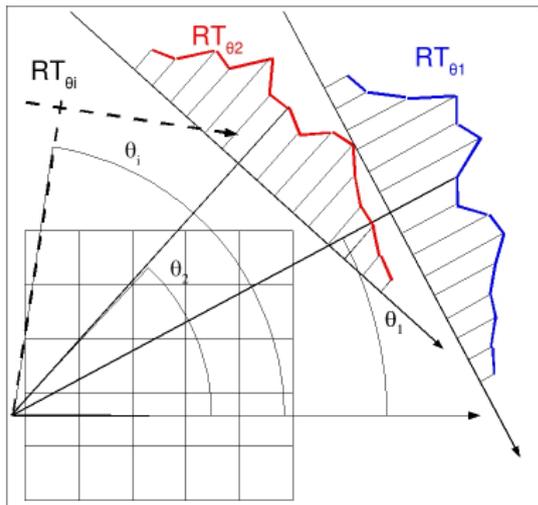
- $RT_{\theta}(\rho)$ is a projection value
- A set of N_{ρ} is the projection RT_{θ} following θ :

$$RT_{\theta} = \{RT_{\theta}(\rho_1), RT_{\theta}(\rho_2), \dots, RT_{\theta}(\rho_{N_{\rho}})\}$$

Discrete Radon Transform and Retroprojection

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$S (N_{\theta} = 180, N_{\rho} = 363)$



from Radon Transform to Sinogram

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- Exemple from Shepp-Logan Fantom image [L. Shepp and B. Logan, 1974]

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Reconstruction :

- Acts as a low-pass filter



BFP :

- Filtered projections
- Increase artefacts

from Radon Transform to Sinogram

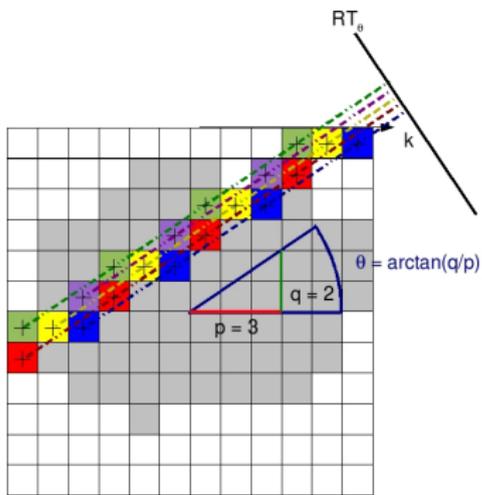
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- Exemple from Shepp-Logan Fantom image [L. Shepp and B. Logan, 1974]
- Retroprojection Results

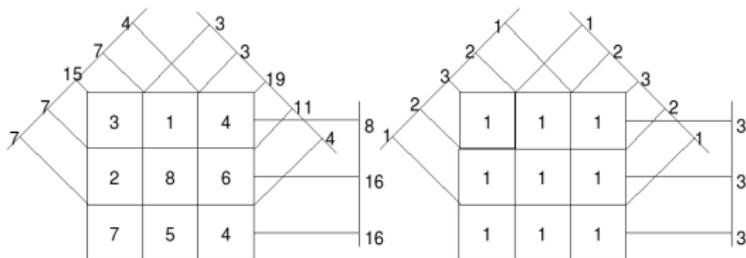


Mojette Transform

- Discrete angles : $(p, q) : \theta = \arctan \frac{q}{p}$
- $GCD(p, q) = 1, q > 0$ unless $p = 1$ and $q = 0$
- On a projection line : only pixels crossed by their center

Definition

$$\mathcal{M}oj_{(p,q)}(b) = \sum_{-\infty}^{+\infty} \sum_{-\infty}^{+\infty} \mathcal{I}(k, l) \Delta(b - kp + lq)$$

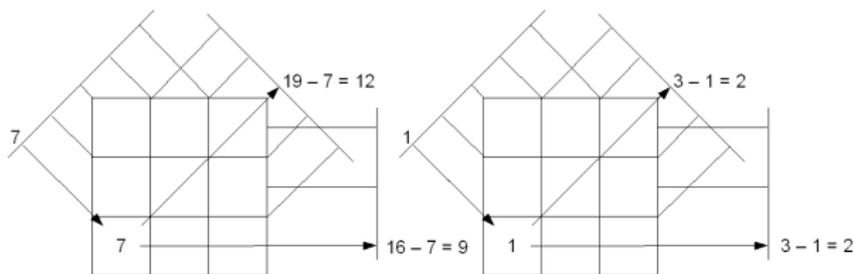


Remarks

- $d\theta$ not constant
- N_ρ and $d\rho$ depend on (p, q)
- Unary sinogram

Iterative Principe of Mojette backprojection

- Each pixel is considered only one time on each projection
- Univoque correspondence between a bin and a pixel
- Pixel reconstructed
- Pixel value subtracted on each projection



Remarks

- Exact Reconstruction (if $\mathcal{P} = \{(p_i, q_i), i \in \mathbb{N}\}$ verify Katz criterion [Katz, 1969])
- $\mathcal{P} = \{(p_i, q_i), i \in \mathbb{N}\}$ can be computed by the Farey series.

Differences between Radon and Mojette

The Radon Transform :

- same N_ρ on each projection, and same sampling δ_ρ
- Constant angle step δ_θ
- Ideal in CT scan, but approximative in discrete domain

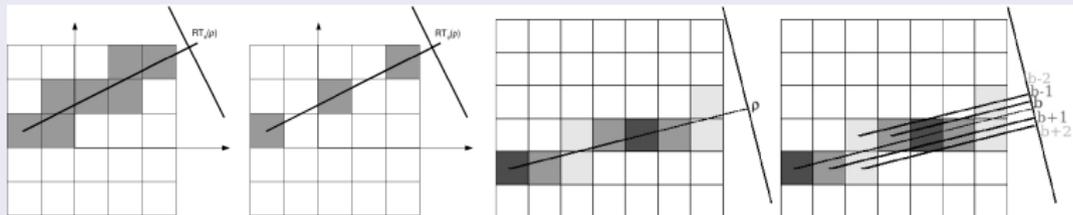
The Mojette Transform :

- Number of bins depends on (p, q)
- δ_θ variable
- Exact reconstruction in discrete domain

Radon sinogram compatible with Mojette backprojection

- 1 : Projection angles in Radon acquisition \Leftrightarrow Mojette angles
- 2 : N_ρ and δ_ρ in Radon acquisition depends on (p, q)
- The RT acquisition is piloted by a set of couples (p, q) (Farey series)

Line decomposition



$RFT_{\theta}(\rho)$ can be expressed as a linear combination of Mojette bin values

$$RFT_{\theta}(\rho) = \sum_{i=0}^{N\rho} B_{p,q}^0(b_i, b) \cdot Moj_{p,q}(b_i)$$

$\theta = \arctan \frac{q}{p}$ and $b \Leftrightarrow \rho$ and $B_{p,q}^0(b_i, b) =$ coefficient between two bins b_i and b

Linear System

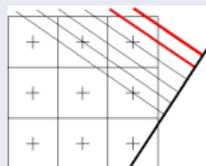
$$\begin{cases} RFT_{\theta}(1) = \sum_{i=0}^{N\rho} B_{p,q}^0(b_i, 1) \cdot Moj_{p,q}(b_i) \\ \dots \\ RFT_{\theta}(N\rho) = \sum_{i=0}^{N\rho} B_{p,q}^0(b_i, N\rho) \cdot Moj_{p,q}(b_i) \end{cases}$$

- $RFT_{\theta}(\rho_i)$: value of sample ρ_i on projection $\theta = \arctan \frac{q}{p}$ acquired with RFT
- $Moj_{p,q}(b_i)$: Mojette bin b_i seeked value

Resolution not assured

- N_ρ seeked values ($Moj_{p,q}(b_i)$)
- N_ρ acquired values
- But some cases give infinity of solutions

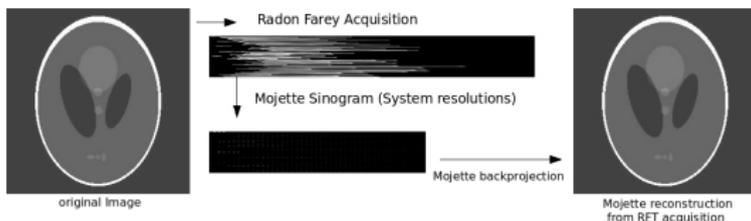
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$$n(p, q) = \lceil \frac{(|p| - |q|) \cdot \sqrt{p^2 + q^2}}{2p} \rceil$$

Triangular System

$$\left\{ \begin{array}{l} Moj_{p,q}(0) = \frac{RFT_\theta(-n)}{\alpha_{p,q}(-n,0)} \\ Moj_{p,q}(1) = \frac{RFT_\theta(-n+1) - Moj_{p,q}(0) \cdot \alpha_{p,q}(-n+1,0)}{\alpha_{p,q}(-n+1,1)} \\ \dots \\ Moj_{p,q}(j) = \frac{RFT_\theta(j-n) - \sum_{k=j-2n \geq 0}^{k < j} Moj_{p,q}(k) \cdot \alpha_{p,q}(j,k)}{\alpha_{p,q}(j-n,j)} \end{array} \right.$$



Exact Reconstruction but not adapted to a real acquisition (CT scan)

- Possible to modify angle step δ_θ between each projection
- Impossible to change N_ρ on a projection

Our purpose :

- N_ρ constant for the acquisition
- Interpolations to recover *RFT* projections from a “Uniform” *RFT* acquisition

Remarks

- (*odd, odd*) projections always give false interpolation results
- For all (*even, odd*) and (*odd, even*) projections :

$\exists N_0, \forall N_\rho \geq N_0 = 2\sqrt{2} \max\{N_\rho(p_i, q_i), (p_i, q_i) \in \mathcal{P}\}$ all interpolations are exact

- In a real acquisition line (discrete pixels \neq continuous data) ?