Slope Fidelity in Terrains with Higher-Order Delaunay Triangulations

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ABSTRACT

Terrains are often modeled by triangulations and one of the criteria is that: triangulation should have "nice shape". Delaunay triangulation is a good way to formalize nice shape. Another criterion is slope fidelity in terrains. In natural terrains there are no abrupt changes in slope. A triangulation for a terrain should use triangles of nice shape and have slope fidelity. To achieve these characteristics, higher-order Delaunay triangulations are used. Two methods are presented and the result of implementations and visualizations show they perform very well on real-world data.

Keywords

Realistic terrains, higher-order Delaunay triangulations, slope fidelity.

1. INTRODUCTION

Terrains are often modeled by triangulations. In nearly all applications where triangulations are used, the triangles must have "nice shape". This is true for scientific data visualization [Att01], mesh generation [She99], computer graphics [Tek00], and terrain modeling [Kok07]. Delaunay triangulation (DT) is a good way to formalize nice shape. Delaunay triangulation of a set *P* of points maximizes the minimum angle of its triangles, over all possible triangulations of *P*, and moreover maximizes lexicographically the increasing sequence [Ber00] of these angles.

For terrain modeling, there are some criteria other than nice shape that a triangulation should have, such as: slope fidelity and drainage reality in terrains [Kre07]. Achieving high slope fidelity is important for terrain-based applications. This paper does provide a reasonable way to fulfill this task for TIN (Triangulated Irregular Network) terrains. Slope fidelity in terrains means that there are no abrupt changes in slope (except at known, specified break lines of the surface such as valleys and ridges) [Kre07]. A triangulation for a terrain should use triangles of nice shape and have slope fidelity. To achieve these two criteria *higher-order Delaunay triangulations* (HODT) [Gud02] are used.

Definition 1 *A triangle in a point set P is order-k if its circumcircle contains at most k points of P. A triangulation of a set P of points is an order-k Delaunay triangulation if every triangle of the triangulation is order-k* (see Fig. 1).

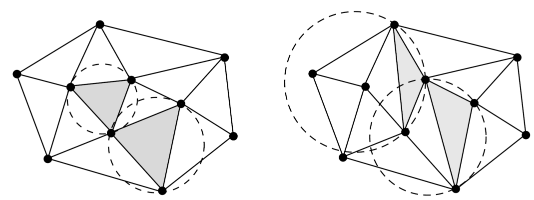

Figure 1. Left, an order-0 Delaunay triangulation. Right, an order-2 Delaunay triangulation, with two triangles of orders 1 and 2.

So a standard Delaunay triangulation is a unique order-0 Delaunay triangulation. For any positive integer *k*, there can be many different order-*k* Delaunay triangulations. By the definition, any order-*k* Delaunay triangulation is also an order-*k'* Delaunay triangulation for all *k' > k* [Gud05]. The higher k, the more freedom to flip the edges, but the shape of triangles may become worse. Higher order Delaunay triangulations also have applications in visualization, mesh generation, computer graphics and realistic terrain modeling [Kok07].

Slope is the most important measure to classify landforms in a terrain. Slope is a property of a plane tangent to a point on a surface. The value for slope at each point of the terrain is usually divided into *gradient* (β), i.e. the steepness of the slope, and *aspect* (ψ), the cardinal direction in which the slope faces [Rei06]. Gradient and aspect are two primary topographic attributes [Spe74] that can be easily estimated using computer-based methods [Moo91].

In natural terrains, slope seems to be quite consistent. When using triangulations for terrain modeling, one should realize that the slope not changed suddenly except for at known, specified break lines of the surface. The DT is a structure defined for a planar set of points, and does not take into account the third dimension at all [Gud02]. In terrain modeling, points have a third coordinate that must be taken into account. Slope inconsistency is an artifact of 3D triangulation. Therefore, minimizing the suddenly changes in slope—maximizing the planarity of terrain surface—is an optimization criterion for terrain modeling.

This paper discusses slope fidelity in terrains using higher-order Delaunay triangulations of a point set P, for which elevations are given. Sect. 2 formalizes the concept of slope and slope fidelity on TINs. To reduce the slope difference between two adjacent triangles and preserve the overall slope consistency, an optimization technique is iteratively applied by flipping the diagonal of a convex quadrilateral that meets a given conditions. This optimization can be performed on a per-quadrilateral wise or on a local neighborhood around a triangle, yielding two variations for TIN terrains: the *elevation* and *local improvement* methods. Sections 3 and 4 cover the elevation and local improvement methods respectively. Sect. 5 gives several experimental results on various terrains. Visualizations show how well, the two proposed algorithms perform on real-world data. Finally the conclusion is given in Sect. 6.

2. SLOPE FIDELITY ON TIN

This section formalizes the concepts of slope and slope fidelity on a TIN. Tajchman [Taj81] represented the surface of a triangle by a plane (*supporting plane*). The equation of a plane determined by three points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$ is $ax+by+cz+d=0$, where the constants a, b, c, and d are determined by simultaneous solution of the equation at P_1, P_2, and P_3. Moore et al. [Moo91] defined the plane gradient, as the intersecting angle of this plane with the horizontal plane (i.e. $z = 0$) by:

$$\beta = \arctan(\sqrt{a^2 + b^2})$$

and the aspect of the plane, measured in degrees clockwise from north, is determined by:

$$\psi = 180 - \arctan\left(\frac{b}{a}\right) + 90 * sign(a)$$

where x is positive east and y is positive north.

The angle between two intersecting planes called *dihedral angle*. The value of the dihedral angle between two planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ is the angle between their normal vectors $\overrightarrow{N_1}(a_1, b_1, c_1)$ and $\overrightarrow{N_2}(a_2, b_2, c_2)$, and can be computed as below:

$$\theta = \arccos\left(\frac{a_1a_2 + b_1b_2 + c_1c_2}{|\overrightarrow{N_1}| + |\overrightarrow{N_2}|}\right).$$

Moet et al. [Moe06] discusses some assumptions that a polyhedral terrain, T, should have to be *realistic*. One assumption which is necessary to bound the shortest path between two points on the terrain is that: the dihedral angle of the supporting plane of any triangle in T with the xy-plane is at most $\pi/2$. It implies that the maximum slope of a line segment on any triangle of T is $\tan(\beta) = O(1)$. So, T becomes more realistic if dihedral angles of its triangles become wider. Small dihedral angles led to an unrealistic polyhedral terrain.

When DT is used for planar set of points, generates a flat surface with dihedral angles of π. If each sample point is lifted to its correct height, and thereby every triangle in the planar triangulation is mapped to a triangle in 3-space, a polyhedral terrain with smaller dihedral angles (less than π in many cases) is generated. It defines a continuous terrain mapped to a piecewise linear interpolation function that is not differentiable at the edges and vertices. Slope becomes inconsistent at these places. Slope inconsistency is reversely related to the terrain dihedral angles; as the dihedral angles become wider and approach π, the surface becomes plainer and more consistent. To preserve the slope consistency in terrain and reduce the slope difference between two adjacent triangles, we must increase their dihedral angle, except at known specified break lines of the surface. Wider dihedral angles lead to plainer surface and vice versa. In fact, we make the interpolation function closer to a differentiable function.

Now, we explain the proposed methods in more details. To implement these methods efficiently, we maintain the set of all convex quadrilaterals in the current triangulation, with some other information such as the order of the two triangles that would be created if the diagonal were flipped. We update the set of convex quadrilaterals and some information after a flip. At most four convex quadrilaterals are deleted and at most four new ones are created by the flip. The order of new incident triangles can be found in $O(\log n+k)$ time, using order-$k+1$ Voronoi diagram [Ram99] after $O(nk\log n)$ preprocessing time.

3. THE ELEVATION METHOD

Given a value of k, the elevation method repeatedly tests whether the diagonal of a convex quadrilateral in the triangulation can be flipped. It will be flipped if two conditions hold simultaneously: (i) the two new triangles are order-k Delaunay triangles. (ii) The elevation difference between new edge endpoints is smaller than the difference between previous edge endpoints. This method connects the co elevation vertices together. Edges become more horizontally, especially in the valleys; this causes a plainer surface. In the nature, water flows in valleys and generates horizontal lines on the valley surface. This method establishes this fact efficiently (See the visualizations of Sect. 5).

The algorithm starts with the Delaunay triangulation and $k' = 1$, then does all flips possible to obtain an order-k' Delaunay triangulation, then increments k' and repeats. This continues until $k' = k$. We first deal with the maximum number of flips needed, and then we discuss the efficiency of the heuristic.

Theorem 1 *The evaluation method terminates after at most $O(n^2)$ flips.*

Proof: Normalize the heights of the vertices to be integers in the range $1, \ldots, n$. Observe that this does not influence the flipping criterion. Consider the function $F(T)$ for a triangulation T:

$$F(T) = \sum_{\overline{uv} \in T} dif(u,v)$$

Where $dif(u,v)$ denotes the difference between elevations of points u and v. Any flip decreases $F(T)$ with at least one, and $F(T)$ is at most $O(n^2)$ to begin with. □

It is obvious that any edge which is flipped out of the triangulation can not reappear. There are at most $O(nk)$ pairs of points in a point set of n points that give order-k Delaunay edges and it takes $O(nk^2+n\log n)$ expected time overall to determine all *useful* order-k Delaunay edges [Gud02]. Thus, this method performs at most $O(nk)$ flips, and it takes $O(k+\log n)$ time per flip to compute the order of new triangles. So, the running time of the elevation method is $O(nk^2+nk\log n)$.

4. THE LOCAL IMPROVEMENT METHOD

Reinbacher et al. [Rei06] introduce the concept of *local gradient* and *local aspect* for each point of the terrain. The local gradient for a point p is defined as a disk on xy-plane with some prespecified radius r, centered at p. The local gradient value to be defined at any point of the terrain will be given by a function $F_g(x,y) = R^2 \to R$ whereas the standard gradient value need not be defined at the edges and vertices. Their definitions led to continuously changing value of the local gradient value at any point on the TIN. Continuity is important for the generation of isogradients [Rei06].

The local improvement method to increase the slope continuity and generating a smoother terrain uses the local neighborhood of triangles. It does not directly use the standard gradient and aspect neither the local gradient nor local aspect. It uses the dihedral angles of triangles. This method repeatedly tests whether the diagonal of a convex quadrilateral in the triangulation can be flipped. For each convex quadrilateral in the triangulation we call it and its four neighbor triangles, *locally adjacent triangles* or *butterfly zone*; usually their projection on the plane is like a butterfly. Figure 2 shows that a butterfly zone consists of six triangles with five dihedral angles incident on the edges $\overline{ab}$, $\overline{ac}$, $\overline{ad}$, $\overline{bc}$ and $\overline{bd}$. There are many choices possible when to allow a flip and when not. For each convex quadrilateral, we check the five dihedral angles of its butterfly zone and flip its diagonal if some conditions hold. This diagonal flipping replaces the dihedral angle at $\overline{ab}$, with another dihedral angle at $\overline{cd}$. It also changes the dihedral angles at the edges $\overline{ac}$, $\overline{ad}$, $\overline{bc}$ and $\overline{bd}$. We decide to flip the diagonal of a convex quadrilateral if the two new triangles are order-k Delaunay and the five new dihedral angles of its butterfly zone are increased; at least 0.2° on average.

Note 1 *Let T be a triangulation with an edge e. Let T' be the triangulation obtained from T by flipping e, which has bigger dihedral angles. Then T' is more smooth than T.*

There are at most $O(nk)$ pairs of points in a point set of n points that give order-k Delaunay edges [Gud02].

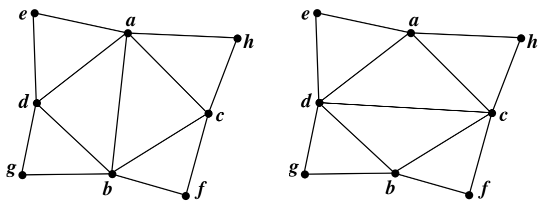

Figure 2. Left: a convex quadrilateral and its butterfly zone, Right: flipping the diagonal of convex quadrilateral.

Theorem 2 *The local improvement method terminates after at most O(n) flips.*

Proof: Let t_1 and t_2 be two adjacent triangles in a triangulation *T*, and let $da(t_1, t_2)$ denotes the dihedral angle between their supporting planes. It is clear that $da(t_1, t_2)$ is in the range of 0 to 180 degree. Consider the function *F(T)*:

$$F(T) = \sum_{t_1, t_2 \in T} da(t_1, t_2)$$

Any flip increases *F(T)* with at least one degree (on average 0.2 degree for any of five triangle pairs), and there are at most *O(n)* pair of adjacent triangles [Ber00]. *F(T)* is at least *O(1)* to begin with, and finally it becomes at most *O(n)*. □

By using order-*k+1* Voronoi diagrams [8] for circular range counting queries, the following theorem can be concluded:

Theorem 3 *The local improvement method to generate smooth terrain in order-k Delaunay triangulations on n points takes O(nk+n*log*n) time after O(nk*log*n) preprocessing time.*

5. EMIRICAL RESULTS AND COMPARISONS

Both elevation and local improvement methods are implemented in C++ and compared with data of five different types of real-world terrains: California hot springs (CHS), Wren peak (WP), Quinn peak (QP), Sphinx lakes (SL) and Split Mountains (SM). The terrains have roughly 1950 vertices. The vertices were chosen by random sampling 1% of the points from elevation grids. Experiments show that the orders from 4 to 8 are more significant; higher orders are less interesting in practice since the interpolation quality may be less good, and skinny triangles may cause artifacts in visualization [Kok07]. The lower orders are also not interesting because they limit our freedom to flip edges. When vertices have the same height, they are treated as a lexicographic number *(z,x,y)*, where *x* and *y* are the lesser significant components in the lexicographic order. We evaluate two important factors for both methods for different values of *k*: (i) the average of all dihedral angles in triangulation (*ada*) and, (ii) the average of minimum angle of all triangles in the plane (*ama*). The results of this experiment are shown in Tables 1 and 2 for CHS, QP and SL terrains respectively (notice that column for *k = 0*, is the output of standard DT). For the other terrains, we got similar results. Generally, increase in *k*, increases the *ada* (smoother surface) and decreases the *ama* (weaker triangulation). In fact there is a trade off between *ada* and *ama.* The local improvement method generates a wider dihedral angles and a smoother terrain, even though the elevation method generates a better triangulation.

Table 1. The ada (in degree) for elevation/local improvement methods to achieve *k*-order DT.

k	0	1	2	4	6	8	13	20
CHS	130/130	133/134	135/136	136/138	137/139	138/140	140/142	142/145
QP	134/134	137/138	139/140	141/142	142/144	143/144	145/146	146/148
SL	132/132	135/136	137/138	139/140	141/142	142/143	143/145	145/147

Table 2. The ama (in degree) for elevation/local improvement methods to achieve *k*-order DT in plane.

k	0	1	2	4	6	8	13	20
CHS	41/41	40/39	39/38	38/37	37/36	37/36	36/35	35/34
QP	42/42	40/40	39/39	38/38	37/37	37/36	36/35	35/34
SL	41/41	39/39	38/38	38/37	37/36	36/35	34/33	33/32

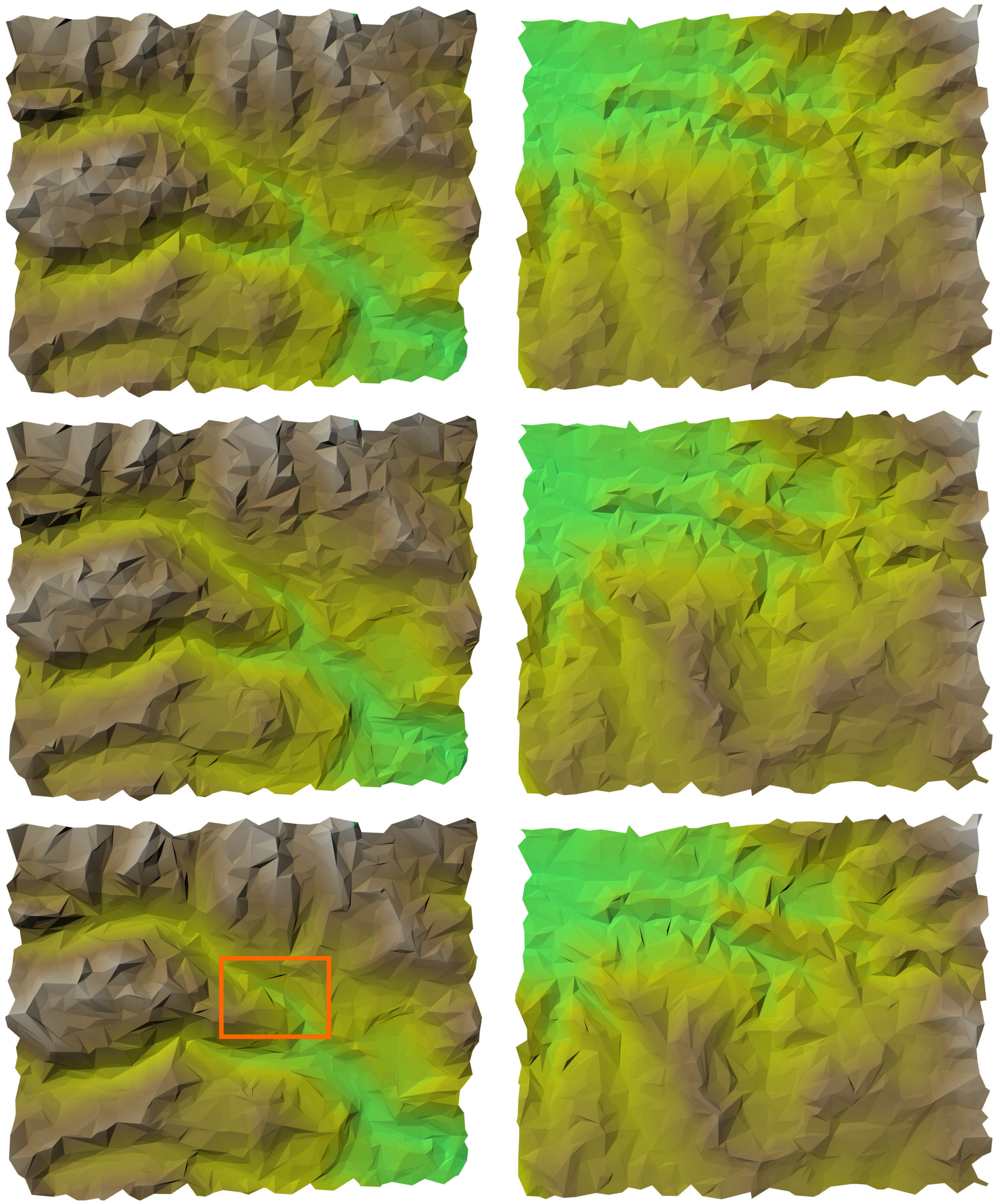

Figure 3. Left: visualizations for Delaunay triangulation (up), elevation (middle) and local improvement (bottom) methods on Sphinx lakes terrain. Right: same, but on Quinn peak terrain.

Figure 3 shows the visualizations that are results of Delaunay triangulation (up), elevation method (middle), and the local improvement method (bottom), on Sphinx lakes and Quinn peak for $k = 8$. When we look at Figure 3 from high above, the three images almost look the same. But when we look at them from a close distance, their differences are quite obvious. Figure 4 shows the shaded area (part of a valley) of figure 3 in more details. It is clear that the generated output from the two proposed methods is more realistic than the output of DT.

The outcome of the elevation method is more realistic than outcome of Delaunay triangulation, and the outcome of local improvement method is better than all of them. The elevation method is very good for modeling valleys and ridges, whereas the local improvement method is good for overall surface and especially for plain surfaces.

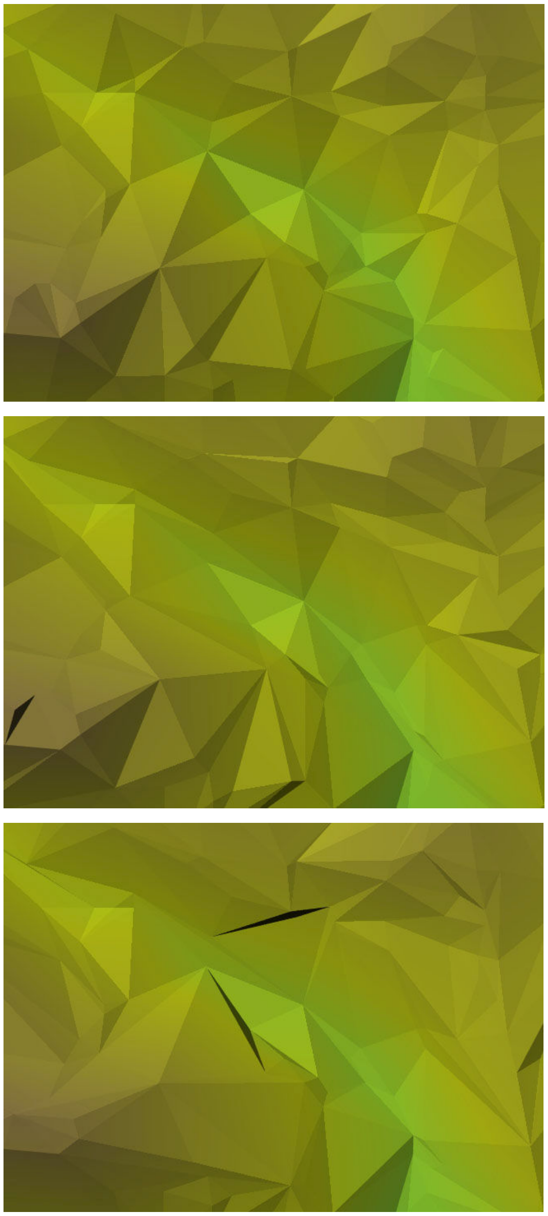

Figure 4. Closer view for DT (up), elevation (middle) and local improvement (bottom).

6. CONCLUSIONS

This paper addresses the issue of slope fidelity in terrains with higher order Delaunay triangulations, used in geo-simulation and visualization contexts. Slope fidelity means that there are no abrupt changes in slope. In natural terrains, slope seems to be quite consistent. When a terrain is modeled by triangulations, it becomes inconsistent. Slope continuity is important criteria for terrain modeling. We present two methods (*elevation* and *local improvement*) for preserving slope consistency in terrain. The results of implementations and visualizations show they perform very well on real world terrains.

Directions for future research include combination of these two methods and investigate them when the valleys and ridges are known in advance. Furthermore, it is interesting to develop similar methods as investigated in this paper to work directly on gradient and aspect.

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