

Stable Integration of the Dynamic Cosserat Equations with Application to Hair Modeling

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Motivation and Contribution

- **Slender structures**
 - Tubes, cords, catheters, hair fibers, DNA strands, drill strings, etc.
- High tensile stiffness and torsion → **stiff equation system**
- Current deformation paradigms (ad-hoc):
 - Mass-spring systems, Linked chains of rigid bodies, Featherstone's Algorithm, FEM, etc.

Mechanical framework

Special Theory of Cosserat Rods [Ant95]

Special Theory of Cosserat Rods

- Mechanical framework for description of spatio-temporal evolution of slender structures
- Director theory
- Accurate deformation model: **bend, twist, shear, extension**
- Complete in theory
- Extensible: contact handling, non-linear material laws
- Set of non-linear coupled PDEs form a BVP
 - BVPs are a domain of shooting or relaxation techniques
- We use a direct approach to this BVP → **relaxation**

How to deal with Cosserat Equations?

- **Currently two models:**

Super-Helices [BAC06], Cosserat Rod Elements [ST07]

- Approximate solutions by means of simple LAGRANGE Mechanics → Energy formulation (variational approach)

- **What do we have to offer?**

- Direct approach to BVP, **no reformulation!**
- New integration technique from structural engineering

Generalized α -Method

Newmark-like implicit integrator of Chung and Hulbert [CH93]

- 1 Unconditionally stable
- 2 Second order accurate
- 3 Controllable numerical damping

Why and when is it important?

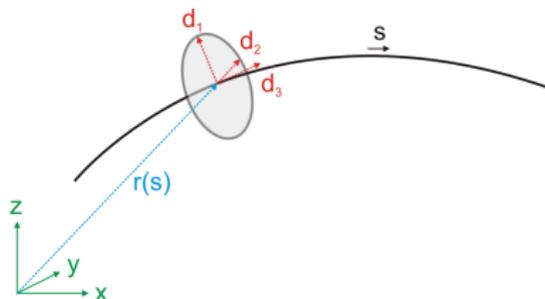
- Certain fields of application require accurate models
 - E.g., medical applications: surgery simulator must be highly reliable ...
- Bouncy models give wrong feed-back on haptic-device



SPECIAL THEORY OF COSSERAT RODS

Geometry

- **Centerline of the rod:** smooth space curve of length L ,
 $\mathbf{r}(s, t) : [s_1, s_2] \times \mathbb{R} \mapsto \mathbb{R}^3$
- $\{\mathbf{d}_i(s, t)\}$ set of **orthogonal directors** furnishing space curve, $\mathbf{d}_1, \mathbf{d}_2$ span the cross section plane, $\partial_s \mathbf{r} \neq \mathbf{d}_3$
- **Configuration** at any time t uniquely determined by
 $\mathcal{C}(s, \cdot) = \{\mathbf{r}(s, \cdot), \mathbf{d}_1(s, \cdot), \mathbf{d}_2(s, \cdot)\}$



Kinematics

- **Kinematics of the directors:**

$$\partial_s \mathbf{d}_k = \boldsymbol{\kappa} \times \mathbf{d}_k, \quad (1)$$

$$\partial_t \mathbf{d}_k = \boldsymbol{\omega} \times \mathbf{d}_k. \quad (2)$$

- $\boldsymbol{\kappa}$ is the Darboux, $\boldsymbol{\omega}$ the twist vector
- κ_1, κ_2 measure bending, κ_3 twist about the three directors

- **Kinematics of the center line:**

$$\partial_s \mathbf{r} = \mathbf{v}. \quad (3)$$

- v_1 and v_2 measure shear, v_3 extension

Compatibility of Darboux and Twist Vector

- **Compatibility** of the local basis $\{\mathbf{d}_k\}$,

$$\partial_t \partial_s \mathbf{d}_k(s, t) = \partial_s \partial_t \mathbf{d}_k(s, t), \quad (4)$$

- gives **compatibility equation**:

$$\partial_s \boldsymbol{\omega} = \partial_t \boldsymbol{\kappa} + \boldsymbol{\kappa} \times \boldsymbol{\omega}. \quad (5)$$

Balance Laws

- **Conservation of linear and angular momentum** leads to the dynamic Cosserat equations:

$$\partial_s \mathbf{n} + \mathbf{f} = \rho A \partial_{tt} \mathbf{r}, \quad (6)$$

$$\partial_s \mathbf{m} + \partial_s \mathbf{r} \times \mathbf{n} + \mathbf{l} = \rho \partial_t (\mathbf{l} \boldsymbol{\omega}), \quad (7)$$

- **n, m**: contact force and contact couple at cross sectional area
- **f, l**: external forces and moments acting on the rod
- **l**: moment of inertia tensor, **A**: area of the cross section, ρ : the linear density

Inextensibility - Infinite Stiffness

- Assume **unshearable** and **inextensible**, set $v = (0, 0, 1)$
- Tangent $\partial_s \mathbf{r}$ now coincides with \mathbf{d}_3 ($\partial_s \mathbf{r} = \mathbf{d}_3$)
- From continuity we get

$$\partial_t \mathbf{d}_3 = \partial_s \mathbf{u}. \quad (8)$$

- Explicit enforcement by ODE
- Acts like infinite stiff spring, introduces severe stiffness into system

Material Laws

- Relate strain variables κ and v to material internal forces \mathbf{n} and torques \mathbf{m}

$$\mathbf{m}(s, t) = \hat{\mathbf{m}}(\kappa(s, t), v(s, t), s), \quad (9)$$

$$\mathbf{n}(s, t) = \hat{\mathbf{n}}(\kappa(s, t), v(s, t), s). \quad (10)$$

- Rod is unshearable and inextensible (i.e. $v = (0, 0, 1)$), material law for \mathbf{m}

$$\mathbf{m}(s, t) = \mathbf{K}(s) \Delta\kappa(s, t), \quad (11)$$

Problem Reduction and Decoupling

- Decomposing all the equations with respect to the local basis $\{\mathbf{d}_k\}$, \rightarrow final system of PDEs

$$\rho A \partial_t \mathbf{u} = \partial_s \mathbf{n} + \boldsymbol{\kappa} \times \mathbf{n} - \rho A (\boldsymbol{\omega} \times \mathbf{u}) + \mathbf{f}, \quad (12a)$$

$$\rho \mathbf{l} \partial_t \boldsymbol{\omega} = \partial_s \mathbf{m} + \boldsymbol{\kappa} \times \mathbf{m} + \mathbf{d}_3 \times \mathbf{n} - \rho (\boldsymbol{\omega} \times \mathbf{l} \boldsymbol{\omega}) + \mathbf{l}, \quad (12b)$$

$$\partial_t \boldsymbol{\kappa} = \partial_s \boldsymbol{\omega} - \boldsymbol{\omega} \times \boldsymbol{\kappa} \quad (12c)$$

$$0 = \partial_s \mathbf{u} + \boldsymbol{\kappa} \times \mathbf{u} - \boldsymbol{\omega} \times \mathbf{d}_3. \quad (12d)$$

- Standard form of structural dynamics problems

$$\hat{\mathbf{M}} \partial_t \mathbf{x}(s, t) + \hat{\mathbf{K}} \partial_s \mathbf{x}(s, t) + [\mathbf{D} \mathbf{x}(s, t)] + \boldsymbol{\Lambda}(s, t) = \mathbf{0}. \quad (13)$$

State vector $\mathbf{x}(s, t) = \{\mathbf{u}(s, t), \boldsymbol{\omega}(s, t), \boldsymbol{\kappa}(s, t), \mathbf{n}(s, t)\}^T$

Boundary Conditions

- **Completing system of equations** by specifying six boundary conditions at both ends of the rod

$$BC_0 := \{\mathbf{u}(0, \cdot) = F_u(t), \boldsymbol{\omega}(0, \cdot) = F_\omega(t)\}, \quad (14)$$

$$BC_L := \{\boldsymbol{\kappa}(L, \cdot) = \mathbf{K}^{-1} \mathbf{m}(L, \cdot) + \hat{\boldsymbol{\kappa}}(L), \mathbf{n}_L = \mathbf{f}(L, \cdot)\}. \quad (15)$$

- BCs can be moved to anywhere
- Other boundary conditions can be integrated easily

Resume

● What do we have now?

- 1 Kinematic relation
- 2 Balance laws
- 3 Material law
- 4 Compatibility equation
- 5 Inextensibility equation
- 6 Boundary conditions

7 System of coupled non-linear PDEs \rightarrow BVP

Discretization using the Generalized α -Method

- **Temporal discretization:** Semi-discrete form

$$\hat{\mathbf{M}}^{1-\alpha_t} \partial_t \mathbf{x}^{1-\alpha_t} + \hat{\mathbf{K}}^{1-\beta_t} \partial_s \mathbf{x}^{1-\beta_t} + \mathbf{\Lambda}^{1-\beta_t} = \mathbf{0}. \quad (16)$$

- Replacement rule: $\diamond^{1-\varepsilon} := (1 - \varepsilon) \diamond^i + \varepsilon \diamond^{i-1}$

$$\begin{aligned} & \hat{\mathbf{M}} \left[(1 - \alpha_t) \partial_t \mathbf{x}^i + \alpha_t \partial_t \mathbf{x}^{i-1} \right] \\ & + \hat{\mathbf{K}} \left[(1 - \beta_t) \partial_s \mathbf{x}^i + \beta_t \partial_s \mathbf{x}^{i-1} \right] \\ & + \left[(1 - \beta_t) \mathbf{\Lambda}^i + \beta_t \mathbf{\Lambda}^{i-1} \right] = \mathbf{0}. \end{aligned} \quad (17)$$

Spatial Discretization of the System

- **Spatial discretization** (same rule on lower indices):

$$\begin{aligned}
 & \hat{\mathbf{M}} \left\{ (1 - \alpha_t) \left[(1 - \alpha_s) \partial_t \mathbf{x}_j^i + \alpha_s \partial_t \mathbf{x}_{j-1}^i \right] \right. \\
 & \quad \left. + \alpha_t \left[(1 - \alpha_s) \partial_t \mathbf{x}_j^{i-1} + \alpha_s \partial_t \mathbf{x}_{j-1}^{i-1} \right] \right\} \\
 & + \hat{\mathbf{K}} \left\{ (1 - \beta_t) \left[(1 - \beta_s) \partial_s \mathbf{x}_j^i + \beta_s \partial_s \mathbf{x}_{j-1}^i \right] \right. \\
 & \quad \left. + \beta_t \left[(1 - \beta_s) \partial_s \mathbf{x}_j^{i-1} + \beta_s \partial_s \mathbf{x}_{j-1}^{i-1} \right] \right\} \\
 & + \left\{ (1 - \beta_t) \left[(1 - \beta_s) \boldsymbol{\Lambda}_j^i + \beta_s \boldsymbol{\Lambda}_{j-1}^i \right] \right. \\
 & \quad \left. + \beta_t \left[(1 - \beta_s) \boldsymbol{\Lambda}_j^{i-1} + \beta_s \boldsymbol{\Lambda}_{j-1}^{i-1} \right] \right\} = \mathbf{0}.
 \end{aligned} \tag{18}$$

Computing derivatives

- **Replacing derivatives** by trapezoidal rule:

$$\partial_t \mathbf{x}^i = \frac{\mathbf{x}^i - \mathbf{x}^{i-1}}{\gamma_t \Delta t} - \frac{1 - \gamma_t}{\gamma_t} \partial_t \mathbf{x}^{i-1}, \quad (19)$$

$$\partial_s \mathbf{x}_j = \frac{\mathbf{x}_j - \mathbf{x}_{j-1}}{\gamma_s \Delta s} - \frac{1 - \gamma_s}{\gamma_s} \partial_s \mathbf{x}_{j-1}, \quad (20)$$

- Δs spatial step, Δt time step, γ weight factor

Stability and Numerical Damping

- **Von Neumann Stability Analysis:** applying the $G\alpha M$ to Dahlquist's test equation $\partial_t x + kx = 0$
- Stability requires that the largest eigenvalue of the amplification matrix $\rho(\mathbf{A}) = \max |\lambda_{1,2}|$ satisfies $\rho \leq 1$
 - Unconditional stability requires $\alpha \leq 1/2, \beta \leq 1/2, \gamma \geq 1/2$
 - Second order accuracy requires $\alpha - \beta + \gamma = 1/2$
- Spectral radius is a measure of the degree of numerical dissipation.
- $G\alpha M$ achieves optimal ratio between high- and low frequency damping

Choice of Integration Coefficients

- How to choose the integration coefficients α, β, γ
- α, β, γ are controlled by λ^∞ for $k\Delta t \rightarrow \infty$
- Not applicable to our equation system (different from Dahlquist's test equation)
- Problem dependend: Its up to you, but you will get used to it :)

Solving the Equation System

- Solving non-linear EQS $\mathbf{F} : \mathbb{R}^{N \times 12} \mapsto \mathbb{R}^{(N-1) \times 12}$

$$\mathbf{F}(\mathbf{x}) = \{F_1(\mathbf{x}_1, \mathbf{x}_2), \dots, F_{N-1}(\mathbf{x}_{N-1}, \mathbf{x}_N)\} = \mathbf{0} \quad (21)$$

- $N \times 12$ unknowns, $(N - 1) \times 12$ equations, 12 BC
- Classical Newton iteration

$$\tilde{\mathbf{x}}^{k+1} = \tilde{\mathbf{x}}^k + \mu \delta \tilde{\mathbf{x}}, \quad \mathbf{J} \delta \tilde{\mathbf{x}} = -\mathbf{F}(\mathbf{x}^k) \quad (22)$$

- Jacobian matrix \mathbf{J} : sparse diagonal - band width 17+1+17, non diag-dominant, non sym. \rightarrow Gauss elimination for sparse band matrices

Integrating the Kinematic Relations

- **Kinematic relations** are decoupled from governing PDEs
- Integration yields matrix exponential: use Rodrigues formula

$$\Delta \mathbf{R}_{j+1}^i = e^{\tilde{\kappa}_j^i \Delta s}, \quad \mathbf{R}_{j+1}^i = \mathbf{R}_j^i \Delta \mathbf{R}_{j+1}^i, \quad (23)$$

- New **positions** \mathbf{r}_{j+1}^i : direct integration of velocities \mathbf{u}^i , inextensibility constraint ensures that new velocities satisfy constraints on shear and extensibility
- Different to the explicit enforcement by considering spatial twists like in [Pai02].

Implementation Details:

- Direct implementation of the above equations in a programming language like C++ is awkward and error prone
- **Computer algebra program** like Maple or Mathematica can produce optimized source code (C, Java, etc.)
- Analytic expression for the Jacobian

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Data:  $i, \Delta t, \mathbf{x}^{i-1}, \mathbf{r}^{i-1}, \mathbf{R}^{i-1}, \mathbf{f}^i, \mathbf{l}^i$ 
Result:  $\mathbf{x}^i, \mathbf{r}^i, \mathbf{R}^i$ 
1: if  $i = 0$  then /* First time step */
2:    $\{\alpha, \beta, \gamma\} \leftarrow \text{compCoefficients}(\rho^\infty)$ ;
3:    $\mathbf{x}^i \leftarrow \text{updateBC}(\mathbf{x}^i)$ ;
4:    $\partial_s \hat{\mathbf{K}} \leftarrow \text{compDxDs}(\hat{\mathbf{K}}, \partial_s \hat{\mathbf{K}}_0, \Delta s, \gamma)$ ;
5:    $\partial_s \mathbf{x}^i \leftarrow \text{compDxDs}(\mathbf{x}^i, \partial_s \mathbf{x}_0^i, \Delta s, \gamma)$ ;
6:    $\partial_t \mathbf{x}^i \leftarrow -\overline{\mathbf{M}}^{-1}(\overline{\mathbf{K}} \partial_s \mathbf{x}^i + \mathbf{\Lambda}^i)$ ;
7: end
8:  $\mathbf{x}^{i-1} \leftarrow \mathbf{x}^i$ ;  $\partial_s \mathbf{x}^{i-1} \leftarrow \partial_s \mathbf{x}^i$ ;  $\partial_t \mathbf{x}^{i-1} \leftarrow \partial_t \mathbf{x}^i$ ;
9:  $k \leftarrow 0$ ;
10: while  $|\mathbf{F}(\mathbf{x}^i)| \geq \varepsilon$  do
11:    $\tilde{\mathbf{x}}^k \leftarrow \Xi(\mathbf{x}^i)$ ;
12:    $\mathbf{J} \leftarrow \partial \mathbf{F}_m / \partial \tilde{\mathbf{x}}_n^k$ ;
13:    $\delta \tilde{\mathbf{x}} \leftarrow \text{sparseSolve}\{\mathbf{J} \delta \tilde{\mathbf{x}} = -\mathbf{F}(\mathbf{x}^i)\}$ ;
14:    $\tilde{\mathbf{x}}^{k+1} \leftarrow \tilde{\mathbf{x}}^k + \delta \tilde{\mathbf{x}}$ ;
15:    $\mathbf{x}_M^i \leftarrow \Xi^{-1}(\tilde{\mathbf{x}}^{k+1})$ ;
16:    $\mathbf{R}^{k+1} \leftarrow \text{updateOrientations}(\mathbf{x}_M^i, \Delta s)$ ;
17:    $\{\mathbf{f}, \mathbf{l}\}^{k+1} \leftarrow \Delta \mathbf{R}^{k+1} \{\mathbf{f}, \mathbf{l}\}^k$ ;
18:    $\mathbf{x}^i \leftarrow \text{updateBC}(\mathbf{x}_M^i)$ ;
19:    $\partial_t \mathbf{x}^i \leftarrow \text{compDxDt}(\mathbf{x}^i, \mathbf{x}^{i-1}, \Delta t, \gamma)$ ;
20:    $\partial_s \mathbf{x}_0^i \leftarrow -\overline{\mathbf{K}}^{-1}(\overline{\mathbf{M}} \partial_t \mathbf{x}^i + \mathbf{\Lambda}^i)$ ;
21:    $\partial_s \mathbf{x}^i \leftarrow \text{compDxDs}(\mathbf{x}^i, \partial_s \mathbf{x}_0^i, \Delta s, \gamma)$ ;
22:    $k \leftarrow k + 1$ ;
23: end
24:  $\mathbf{r}^i \leftarrow \text{updatePositions}(\mathbf{x}^i, \Delta s)$ ;
25:  $\{\mathbf{f}, \mathbf{l}\}^{i-1} \leftarrow \{\mathbf{f}, \mathbf{l}\}^i$ 

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RESULTS

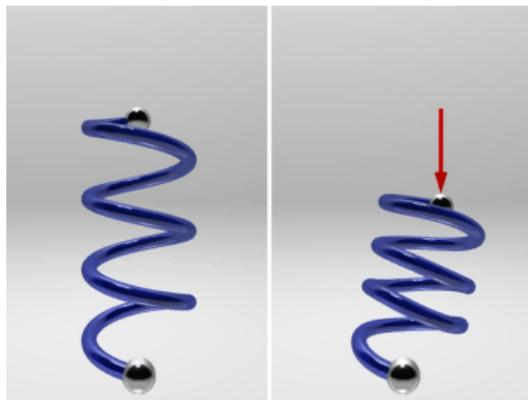
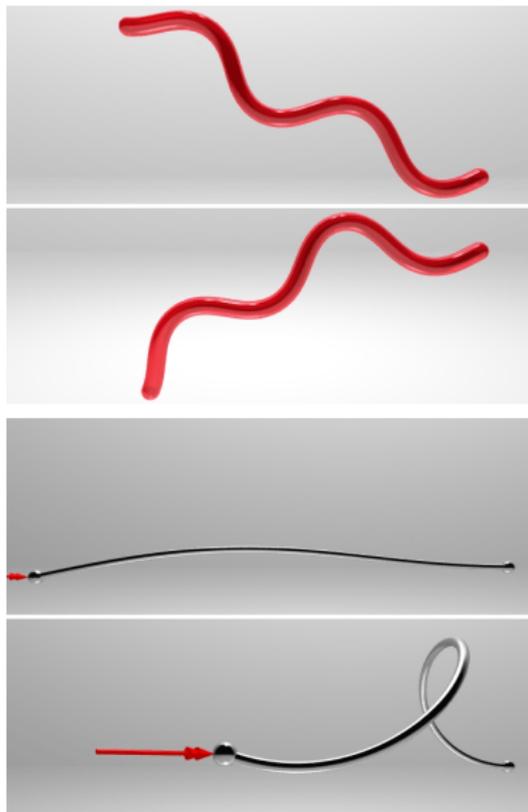


Figure: **Top row:** 1.) Non-straight rod subject to gravity; 2.) Helical rod (high damping) subject to end point load. **Bottom row:** 3.) Straight rod subject to end point torque. Parameters: $\alpha = 0.4$, $\beta = 0$, and $\gamma = 1.0$

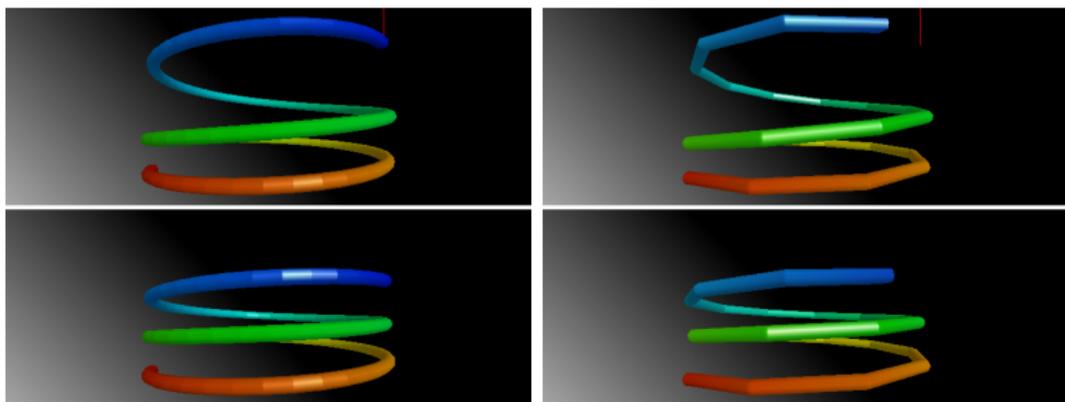


Figure: Coil spring with low damping which is excited by a vertical end point load and released after 0.1 secs (100 and 25 segments).



Figure: The canonical example - buckling.



Figure: Velocity driven extension of a hair tress (25 cm, Keratin, deformable guide, 180 interpolated fibers). Tress is released after a total elongation of 30 % of bounding box length. Boundary conditions: $BC_L := \{\mathbf{v}, \boldsymbol{\omega}\} = \{\lambda \mathbf{d}_{13}, \lambda \mathbf{d}_{23}, \lambda \mathbf{d}_{33}, 0, 0, 0\}$, extension speed $\lambda = -2.0$.

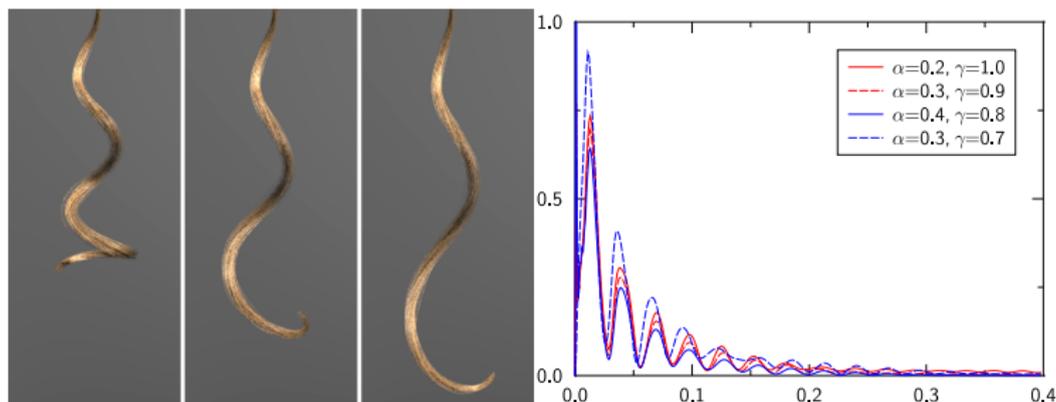


Figure: Hair tress under gravity and its kinetic energy ($\beta = 0$).

All examples: Relaxation procedure converges within 14 iterations on the average. Time until convergence for 100 segments approx. 135 ms (Laptop, Intel Pentium M (Centrino), 1.86 GHz and 1GB RAM). Time step $\Delta t = 1/30$ in all examples, $\alpha = 0.3$, $\beta = 0$, and $\gamma = 0.7$.



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Thank you.