TESTING MONTE-CARLO GLOBAL ILLUMINATION METHODS WITH ANALYTICALLY COMPUTABLE SCENES

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ABSTRACT

The paper presents analytically computable scenes for testing global illumination algorithms with arbitrary BRDFs. The task of these scenes is to enable us to compare global illumination algorithms and check the correctness of the implementation. In our first approach a criterion is given that makes the radiance constant for an arbitrary closed scene allowing either arbitrary BRDFs or arbitrary lightsource models. In the second approach the geometry is assumed to be an internal surface of a sphere. Here homogeneous diffuse and mirror like reflections can be tested with arbitrary lightsource models.

Keywords: Global illumination, BRDF sampling, albedo, test scenes, Monte-Carlo integration, rendering equation

1 Introduction

Testing graphics algorithms is rather difficult since the result is a tone mapped or at least scaled image that can only be subjectively analyzed. If global illumination solution is required, the image contains a lot of non-trivial phenomena, including multiple reflections, caustics, soft shadows, etc. The programmer tends to believe that some artifacts are due to these phenomena instead of looking for further implementation errors. The situation gets even worse when Monte-Carlo methods are applied, since in this case the result is a random variable that forces the developer to explain all computation errors by random noise. To validate these algorithms objectively and measure their convergence, we need test scenes for which the exact solution is known.

Our goal is to test global illumination algorithms that solve the rendering equation without significant simplifications. The rendering equation has the following form

$$L(\vec{x},\omega) = L^e(\vec{x},\omega) + \mathcal{T}L(\vec{x},\omega).$$

Operator $\mathcal{T}$ calculates a single reflection of the radiance function

$$\mathcal{T}L(\vec{x},\omega) = \int_{\Omega'} L^{\rm in}(\vec{x},-\omega') \cdot \cos\theta' \cdot f_r(\vec{x},\omega',\omega)\; d\omega',$$

where Ω' is the hemisphere above point $\vec{x}$, $L(\vec{x},\omega)$ is the radiance of the surface in point $\vec{x}$ at direction ω, $L^{\rm in}(\vec{x},-\omega')$ is the incoming radiance from direction ω', θ' is the angle between the surface normal and direction $-\omega'$, and $f_r(\vec{x},\omega',\omega)$ is the bi-directional reflection/refraction function (figure 1).

Standard set of models in ray tracing were known in the late 80's [3]. These models en-

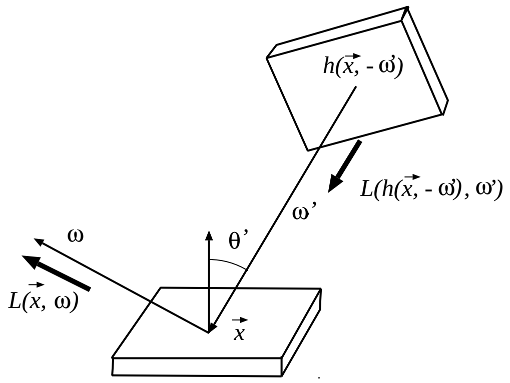

Figure 1: Geometry of the rendering equation

abled to compare the resulting images and the computation speed with other algorithms and determined whether or not a tested algorithm can be accepted. A similar approach was proposed by Ward and Shirley for global illumination algorithms and they also made a set of models for testing [8, 12].

Smits and Jensen [13] introduced another set of test scenes for diffuse and ideally specular surfaces to exercise, compare and validate different global illumination algorithms. The models and the solutions obtained with brutal force path tracing are available on the internet.

Another direction of developing reference scenes is the identification of those combinations of the scene geometry, lightsource intensity and BRDFs, for which the solution can be obtained analytically. An especially important class contains scenes with constant radiance solution, since a constant value can easily be compared with the actual solution.

A well known test scene meeting this requirement is the *furnace* [13, 10, 6] where the environment is closed and every surface has the same diffuse albedo a and diffuse emission L^e. Now the solution at any point is $L(\vec{x}) = L^e/(1-a)$. Note the furnace provides a comfortable scene for gathering type random walk algorithms since the emission is the same in all directions. This does not allow to test many features and results in too good performance measurements. Bekaert et. al. used another version to compare diffuse radiosity algorithms [1]. They assumed that the sum of the diffuse emission and the diffuse albedo are 1 everywhere, which results in a constant 1 solution. In these models the geometry is arbitrary.

Hyben and Ferko [4] proposed a completely different approach first in the flatland then in the 3D space, where the geometry is an internal surface of a circle or a sphere, the surface reflection is constant and uniform and the emission is constant in a subset of the sphere. In this environment the reflected radiance turns out to be constant. Later another proof was given for the 3D case supposing that the lightsource emission is arbitrary diffuse [16] and it was demonstrated that not only the complete solution but also the partial solutions of different bounces are constant. Recently Hyben and Ferko [5] further enhanced their flatland solution and allowed obstacles in the sphere. Doing so, they had to give up the benefit that the radiance can be obtained in closed form, and provided the solution using a function series form with controllable accuracy.

This paper contributes to both directions. It extends the scenes with constant radiance solution and for arbitrary BRDFs and lightsource functions. In order to find analytically solvable scenes, we use reverse approach. We start with a prescribed distribution and search for closed scenes where the radiance would be identical to the given radiance. We shall show that this requirement can be met for arbitrary closed geometry if the local albedo and lightsource intensity satisfies a given relation. We also examine how this criterion can be met. Two special cases are examined. In the first case, called *BRDF testing*, we are supposed to know the BRDF and we are looking for the lightsource which gives constant radiance for arbitrary scene. In the second case, called the *lightsource testing*, we assume that the lightsources are arbitrary and we set the parameters of the surfaces in such a way that the radiance could be determined analytically. The paper also investigates the test scenes of spherical geometry. It proves that in the diffuse case the lightsource function can be arbitrary and it needs not be constant in a subset of the surface, and a new analytically solvable case is presented that requires the surfaces to be ideal mirrors.

2 Scenes with constant radiance

One of the benefits of having analytically computable scenes is that it can help verify that an algorithm is computing correct solution. In these scenes all the algorithmic feature could be kept and they could give easily representable solutions. In our approach we aim at constant radiance solution since in this way the reference can be defined by a single scalar value.

Suppose that the radiance is constant everywhere and at every direction ($L(\vec{x},\omega) = \tilde{L}$) and also that the scene is a *closed environment*, i.e. looking at any direction we can see a surface, thus the incoming radiance is also constant ($L^{\text{in}}(\vec{x},\omega) = \tilde{L}$). All these constant values are substituted into the rendering equation, which expresses the radiance $\tilde{L}$ of point $\vec{x}$ at direction ω as a sum of the emission and the reflection of all point radiances that are visible from here. After the substitution we get:

$$\tilde{L} = L^e(\vec{x},\omega) + (\mathcal{T}\tilde{L})(\vec{x},\omega) =$$

$$L^e(\vec{x},\omega) + \int_\Omega \tilde{L} \cdot f_r(\omega',\vec{x},\omega) \cdot \cos\theta' \, d\omega' =$$

$$L^e(\vec{x},\omega) + \tilde{L} \cdot \int_\Omega f_r(\omega',\vec{x},\omega) \cdot \cos\theta' \, d\omega' =$$

$$L^e(\vec{x},\omega) + \tilde{L} \cdot a(\vec{x},\omega),$$

since the albedo is defined by the equation

$$a(\vec{x},\omega) = \int_\Omega f_r(\omega',\vec{x},\omega) \cdot \cos\theta' \, d\omega'. \qquad (1)$$

Summarizing, if the radiance is $\tilde{L}$ at every point and direction, then the following relation holds:

$$a(\vec{x},\omega) = 1 - \frac{L^e(\vec{x},\omega)}{\tilde{L}}. \qquad (2)$$

Since the solution of the rendering equation is unambiguous if the light transport operator is a contraction, i.e. for close environment the albedo is less than 1, this is a necessary and sufficient requirement for the radiance to be constant. For physically plausible models, the emission function and the albedo cannot be negative, and the albedo cannot exceed 1. This holds if $\tilde{L} > \max L^e(\vec{x},\omega) \geq 0$.

If L^e is constant, then the required albedo will also be constant. Note that this gives back the furnace test scene as a special case. If $\tilde{L}$ is 1 and the albedo is direction independent (i.e. diffuse), then the lightsources are also diffuse and the sum of the albedo and emission are 1. Note that this is the other special case that has already been used in the literature [1]. However, these are not the only alternatives. The BRDFs and the emission function should not even be diffuse in order to meet equation (2). Thus the non-diffuse rendering algorithms can also be tested with this scene. This is, unfortunately, not as simple as for diffuse scenes. If the surface reflection is general, complex BRDF and emission functions may show up, for which the albedo computation and the enforcement of equation (2) get difficult or even impossible. In the next sections this problem is attacked from two directions. First arbitrary BRDFs are allowed and the emission is set accordingly. Then the emission is given total freedom, and the BRDFs are defined to respect equation (2).

2.1 BRDF testing

Suppose that when testing the global illumination algorithm in general we are also interested in validating the BRDF models and their importance sampling features. Thus the material functions are expected not to change during testing, but we are looking for the lightsource which gives constant radiance for arbitrary scenes:

$$L^e(\vec{x},\omega) = \tilde{L} \cdot (1 - a(\vec{x},\omega)). \qquad (3)$$

In practice, materials are described by f_r BRDFs. This formula, on the other hand, requires an emission function that can be computed from the albedo. The albedo, in turn, is the integral of the cosine weighted BRDF. Unfortunately, this integral cannot be evaluated for most of the practical BRDFs, only for some simple ones including the diffuse reflection and the ideal-mirror like reflection [7, 14]. Thus instead of computing this emission analytically, a simulation method is used that provides a random emission function with the mean satisfying this requirement. Note that Monte-Carlo algorithms obtain the solution as an average, thus this random simulation does not distort the result.

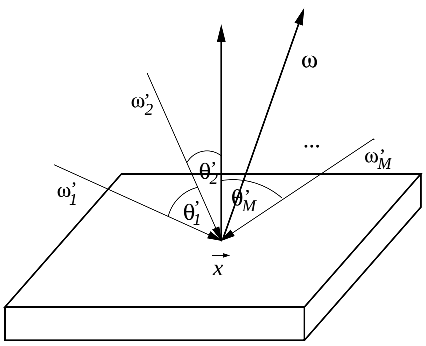

Figure 2: Simulation of the lightsource

Assume that during the global illumination algorithm, the value of $L^e(\vec{x},\omega)$ is needed for a particular surface point $\vec{x}$ and direction ω. The orientation of the surface or the normal vector is also known. In order to find a random estimate for $L^e(\vec{x},\omega)$, the following simulation step is executed: $\omega'_1,\ldots,\omega'_i,\ldots,\omega'_M$ random directions are generated using a uniform distribution on the hemisphere above the surface at $\vec{x}$. The angles θ'_i between these directions and the surface normal, and the BRDFs $f_r(\omega'_i,\vec{x},\omega)$ for these incoming directions and for output direction ω are evaluated. From the BRDF values the emission is estimated by the following formula:

$$\hat{L}^e(\vec{x},\omega) = \tilde{L}\cdot(1-\frac{2\pi}{M}\sum_{i=1}^{M} f_r(\omega'_i,\vec{x},\omega)\cdot\cos\theta'_i). \tag{4}$$

The expected value of this random estimator is:

$$E[\hat{L}^e(\vec{x},\omega)] = \int_\Omega \hat{L}^e(\vec{x},\omega)\cdot\frac{d\omega'}{2\pi}$$

where $1/(2\pi)$ is the probability density of the choice of the incoming directions on the upper hemisphere. Subtituting equation (4) we can prove that the expected value of this emission function really meets requirement (3):

$$E[\hat{L}^e(\vec{x},\omega)] =$$

$$\int_\Omega \tilde{L}\cdot(1-\frac{2\pi}{M}\sum_{i=1}^{M} f_r(\omega'_i,\vec{x},\omega)\cdot\cos\theta'_i)\frac{d\omega'}{2\pi} =$$

$$\tilde{L}\cdot(1-\int_\Omega f_r(\omega',\vec{x},\omega)\cdot\cos\theta'\,d\omega').$$

2.2 Lightsource testing

Let us now suppose that the lightsources are tested together with the general features of the global illumination algorithm, thus the lightsources are not altered and the material models are set to make the radiance constant. It means that given an $L^e(\vec{x},\omega)$ emission function, the BRDF $f_r(\omega,\vec{x},\omega')$ should be found to satisfy the following equation

$$\int_\Omega f_r(\omega',\vec{x},\omega)\cdot\cos\theta'\,d\omega' =$$

$$a(\vec{x},\omega) = 1-\frac{L^e(\vec{x},\omega)}{\tilde{L}}.$$

Note that we have constraints on the integral of the BRDF, i.e. the albedo. There could be a lot BRDFs that can meet this requirement. It could be taken into account that a physically plausible BRDF model is symmetric since this fact is often exploited by global illumination algorithm. Formally, the Helmholtz-symmetry [9] states that the incoming and outgoing directions can be exchanged in the BRDF:

$$f_r(\omega,\vec{x},\omega') = f_r(\omega',\vec{x},\omega). \tag{5}$$

If we are looking for the BRDF in a separable form [11, 2], then we automatically fulfill the requirement of the symmetry. The separability means that the BRDF is a product of two similar functions parameterized by ω and ω' respectively:

$$f_r(\omega',\vec{x},\omega) = f(\vec{x},\omega)\cdot f(\vec{x},\omega'). \tag{6}$$

The albedo is calculated as an integral on the hemisphere, thus we obtain:

$$a(\vec{x},\omega) = \int_{\Omega} f(\vec{x},\omega') \cdot f(\vec{x},\omega) \cdot \cos\theta' \; d\omega' =$$

$$f(\vec{x},\omega) \cdot \int_{\Omega} f(\vec{x},\omega') \cdot \cos\theta' \; d\omega'. \qquad (7)$$

It can be noticed that the value $f(\vec{x},\omega)$ is directly proportional to the value $a(\vec{x},\omega)$, so there exits a parameter λ for which

$$f(\vec{x},\omega) = \lambda \cdot a(\vec{x},\omega).$$

Substituting this into equation (7), we get:

$$a(\vec{x},\omega) = \lambda^2 \cdot a(\vec{x},\omega) \cdot \int_{\Omega} a(\vec{x},\omega') \cdot \cos\theta' \; d\omega'.$$

The unknown parameter λ can be easily determined

$$\lambda^2 = \frac{1}{\int_{\Omega} a(\vec{x},\omega') \cdot \cos\theta' \; d\omega'}. \qquad (8)$$

Now, all the necessary relations are known so the BRDF can be expressed from the albedo and then from the prescribed emission function:

$$f_r(\omega',\vec{x},\omega) = \lambda^2 \cdot a(\vec{x},\omega) \cdot a(\vec{x},\omega') \Longrightarrow$$

$$f_r(\omega',\vec{x},\omega) = \frac{a(\vec{x},\omega) \cdot a(\vec{x},\omega')}{\int_{\Omega} a(\vec{x},\omega') \cdot \cos\theta' \; d\omega'} =$$

$$\frac{(\tilde{L} - L^e(\vec{x},\omega)) \cdot (\tilde{L} - L^e(\vec{x},\omega'))}{\tilde{L} \cdot \int_{\Omega} (\tilde{L} - L^e(\vec{x},\omega')) \cdot \cos\theta' \; d\omega'}. \qquad (9)$$

Thus a correspondence is established between the known emission function and an appropriate BRDF. The denominator of the BRDF formula still contains an integral that should be evaluated. This integral can be written as:

$$\int_{\Omega} (\tilde{L} - L^e(\vec{x},\omega')) \cdot \cos\theta' \; d\omega' =$$

$$\pi\tilde{L} - \int_{\Omega} L^e(\vec{x},\omega') \cdot \cos\theta' \; d\omega'. \qquad (10)$$

Let us first suppose that the total power Φ and the area S of the lightsource is known and the emission intensity is homogeneous for different points in S. In this case the integral in equation (10) is Φ/S, thus the BRDF is as follows:

$$f_r = \frac{(\tilde{L} - L^e(\vec{x},\omega)) \cdot (\tilde{L} - L^e(\vec{x},\omega')) \cdot S}{\tilde{L} \cdot (\pi\tilde{L} \cdot S - \Phi)}.$$

If the total power is not known, then a random simulation can be executed. Whenever the reflection at point $\vec{x}$ is calculated, random directions $\omega'_1, \ldots, \omega'_i, \ldots, \omega'_M$ are generated using a uniform distribution on the hemisphere above the surface at $\vec{x}$. The angles θ'_i between these directions and the surface normal, and the emission values $L^e(\vec{x},\omega'_i)$ for these directions are evaluated. The integral of equation (10) is replaced by the following random value:

$$f_r = \frac{(\tilde{L} - L^e(\vec{x},\omega)) \cdot (\tilde{L} - L^e(\vec{x},\omega'))}{\pi\tilde{L} \cdot (\tilde{L} - \frac{2}{M}\sum_{i=1}^{M} L^e(\vec{x},\omega'_i) \cdot \cos\theta'_i)}.$$

As we did for BRDF sampling, it can be shown that the expected value of this random BRDF function satisfies equation (9).

Monte-Carlo global illumination algorithms often involve BRDF sampling, which means that directions are sampled with a probability density that mimics the cosine weighted BRDF. It is not easy to find an accurate probability density for the constructed BRDF. However, for testing purposes a simple scheme is suitable, for instance, where the probability density is proportional to the cosine of the angle between the surface normal and the generated direction.

2.3 Uniformly sampling directions on a hemisphere

An elementary operation of both BRDF testing and lightsource testing is the generation of uniformly distributed directions on a hemisphere above a surface of normal $\vec{n}$. In this section, for the sake of completeness, we recall an algorithm that can provide this in a simple way. Let us place a Cartesian coordinate system into the center, extend the hemisphere into a sphere and enclose the sphere by a cube $[-1,1]^3$. A uniformly distributed point in this cube can be obtained by letting the random number generator find numbers x, y, z in $[-1,1]$. Then we check whether or not the point defined by the three numbers is above the plane of normal $\vec{n}$ and is inside the unit sphere. If both conditions are met, then the $[x, y, z]$ vector is normalized, that is, it is projected to the surface of the sphere, otherwise new triplets are generated. The code of the algorithm, that uses an rnd() function for obtaining random values in $[0,1]$, is the following:

SampleHemiSphere($\vec{n}$)
    while (true)
        x = 2· rnd()-1
        y = 2· rnd()-1
        z = 2· rnd()-1
        $\vec{d} = [\ x, y, z\ ]$
        if $\vec{d} \cdot \vec{n} > 0$ **then**
            $l = |\vec{d}|$
            if $l < 1$ **then return** $\vec{d}/l$
        endif
    endwhile
end

This algorithm generates a random direction using less number of operations than if the two spherical directions were sampled. If the resulting BRDF turned out to be non-uniform, then a Phong-like sampling would be more appropriate. Such sampling strategies can be found in [7, 14].

3 Spherical reference scene

So far we have supposed that the geometry is arbitrary. Now a special geometry is examined that allows more freedom in setting the BRDFs and the lightsources, but still provides analytical solution. This special geometry is the internal surface of a sphere. Two analytical solutions are given. First, it is assumed that the surfaces and the lightsources are all diffuse, then the surfaces are supposed to be perfect mirrors.

3.1 Diffuse spherical scene

Let the scene be an inner surface S of a sphere of radius R, the BRDF be constant $f_r = a/\pi$ and the emission be diffuse and defined by $L^e(\vec{x})$. Note that unlike in previous papers we do not assume that the emission is constant in a subset of the surface points, but it can follow an arbitrary function. Using the

$$d\omega' = \frac{\cos\theta_{\vec{y}} \cdot d\vec{y}}{|\vec{y} - \vec{x}|^2}$$

substitution for the solid angle, we obtain the following form of the light transport operator:

$$(\mathcal{T}L^e)(\vec{x}, \omega) = \int_S f_r \cdot \cos\theta_{\vec{x}} \cdot L^e(\vec{y}) \cdot \frac{\cos\theta_{\vec{y}}}{|\vec{y} - \vec{x}|^2}\, d\vec{y}. \tag{11}$$

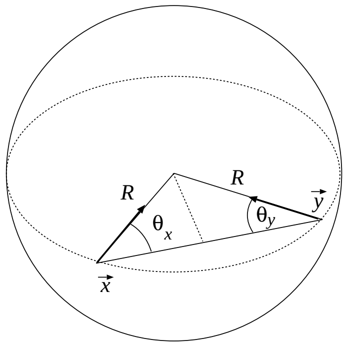

Figure 3: Geometry of the reference scene

Looking at figure 3, we can see that inside a sphere $\cos\theta_{\vec{x}} = \cos\theta_{\vec{y}} = \frac{|\vec{y}-\vec{x}|}{2R}$, thus we can obtain the following final form:

$$(\mathcal{T}L^e)(\vec{x}, \omega) = \int_S f_r \cdot L^e(\vec{y}) \cdot \frac{\cos\theta_{\vec{x}} \cdot \cos\theta_{\vec{y}}}{|\vec{y} - \vec{x}|^2}\, d\vec{y} =$$

$$\frac{f_r}{4R^2} \cdot \int_S L^e(\vec{y})\ d\vec{y} = a \cdot \frac{\int_S L^e(\vec{y})\ d\vec{y}}{4R^2\pi}.$$

The response is equal to the product of the albedo and the average emission of the total spherical surface.

Using this formula recursively, the Neumann series expansion of the solution of the rendering equation is:

$$L(\vec{x}) = L^e(\vec{x}) + \sum_{i=1}^{\infty} (\mathcal{T}^i L^e)(\vec{x}) =$$

$$L^e(\vec{x}) + \frac{\int_S L^e(\vec{y})\ d\vec{y}}{4R^2\pi} \cdot (a + a^2 + \ldots) =$$

$$L^e(\vec{x}) + \frac{\int_S L^e(\vec{y})\, d\vec{y}}{4R^2\pi} \cdot \frac{a}{1-a}.$$

3.2 Mirror sphere scene

Now, let us assume that the surface is a homogeneous ideal mirror which also emits light with intensity L^e. Suppose that the emission intensity is isotropic and similar in all points, i.e. it can be characterized by an $L^e(\theta)$ function where θ is the angle between the emission direction and the surface normal. The reflected emission $(\mathcal{T}L^e)(\vec{x}, \omega)$ is the emission of the point that is visible in the mirror direction of ω from $\vec{x}$. If the outgoing angle between the surface normal at $\vec{x}$ and ω is θ, then the incoming angle between this surface normal and the reflection direction is also θ. Furthermore, the outgoing angle at the visible point is also θ due to the spherical geometry (see figure 3). Thus the outgoing radiance after the reflection is $k_r \cdot L^e(\theta)$ where k_r is the reflection factor. The complete solution can be obtained as a Neumann series:

$$L(\theta) = \sum_{i=1}^{\infty} (\mathcal{T}^i L^e)(\theta) =$$

$$L^e(\theta) \cdot (1 + k_r + k_r^2 \ldots) = \frac{L^e(\theta)}{1 - k_r}.$$

4 Simulation results

In order to demonstrate the testing power of the proposed methods, we have selected a Sierpienski scene and rendered with a stochastic iteration algorithm [15]. Three BRDF/lightsource combinations were considered: diffuse with analytically computed albedo and lightsources, diffuse with simulated lightsource and specular with simulated lightsource. The average albedo in all cases is 0.47. The exponent of the Phong reflection has been set to 10. Recall that the presented algorithm introduces some noise controlled by the number of incoming directions M. This additional noise should be low not to distort that of the tested algorithm. Now the lightsources are simulated with a single random incoming direction ($M = 1$). Note that the error curves using the simulated and the exactly computed lightsources are similar, thus even this high degree of randomization of the lightsource does not add to much noise. The images after 10 iteration steps are shown in figure 5 and the error data in figure 4, respectively. The fully converged images are completely white in all cases.

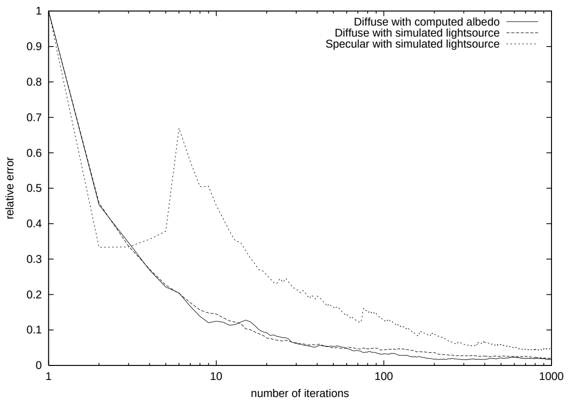

Figure 4: Error curves for the three demonstrated scenes

5 Conclusions and future work

The paper discussed the development of analytically solvable scenes for the purpose of testing Monte-Carlo global illumination algorithms. Unlike other similar approaches, we have not restricted the models only to the diffuse case but also allowed for arbitrary BRDF models and lightsources. Two fundamental approaches have been presented. First we proposed criteria that guarantee that the radiance of a closed scene is constant. Then a special geometry, the internal surface of a sphere was considered. Here we have shown that the reflected radiance can be made constant in a special diffuse setting and also that allowing only ideal reflection the solution can be obtained for arbitrary isotropic lightsource models.

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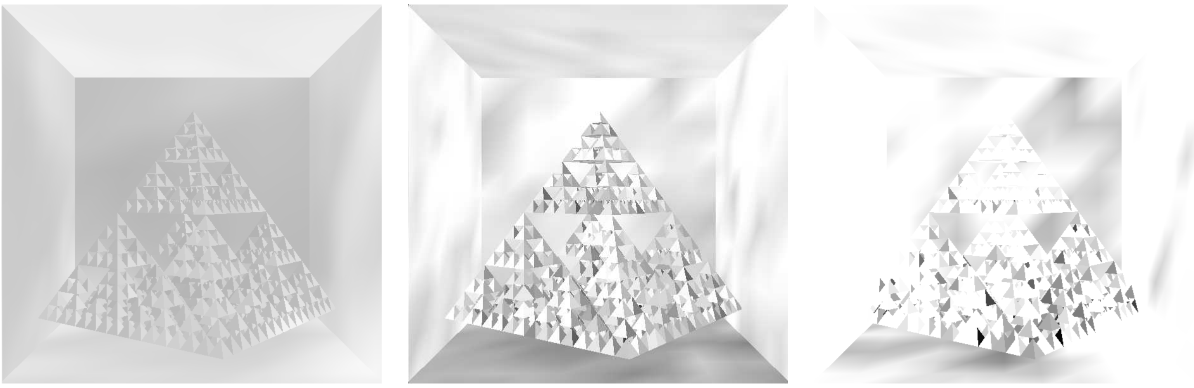

Figure 5: Temporary images after 10 iterations steps for the diffuse scene with computed lightsource (left), the diffuse scene with simulated lightsource (middle), and the specular scene with simulated lightsource(right)

REFERENCES

[1] P. Bekaert, R. Cools, and D. Willems. An empirical comparison of monte carlo radiosity algorithms. In *Winter School of Computer Graphics '99*, pages 9–16, Plzen, Czech Republic, 1999.

[2] A. Fournier. Separating reflection functions for linear radiosity. In *Proceedings of the Sixth Eurographics Workshop on Rendering*, pages 296–305, 1995.

[3] E. A. Haines. A proposal for standard graphics environments. *IEEE Computer Graphics and Applications*, 7(7):3–5, 1987.

[4] M. Hyben, I. Martisovits, and A. Ferko. Scene complexity for rendering in flatland. In L. Szirmay-Kalos, editor, *Spring Conference on Computer Graphics*, pages 112–120, 1998.

[5] M. Hyben, I. Martisovits, A. Ferko, and M. Prochaczka. On the exact solution of rendering equation in flatland. In *Spring Conference of Computer Graphics '00*, pages 222–227, Budmerice, 2000.

[6] A. Keller. A quasi-Monte Carlo algorithm for the global illumination in the radiosity setting. In H. Niederreiter and P. Shiue, editors, *Monte-Carlo and Quasi-Monte Carlo Methods in Scientific Computing*, pages 239–251. Springer, 1995.

[7] E. Lafortune and Y. D. Willems. Using the modified phong reflectance model for physically based rendering. Technical Report RP-CW-197, Department of Computing Science, K.U. Leuven, 1994.

[8] G. W. Larson and R. Shakespeare. *Rendering with Radiance*. Morgan Kaufmann, California, 1998.

[9] M. Minnaert. The reciprocity principle in lunar photometry. *Astrophysical Journal*, 93:403–410, 1941.

[10] P. Shirley. Time complexity of Monte-Carlo radiosity. In *Eurographics '91*, pages 459–466. Elsevier Science Publishers, 1991.

[11] P. Shirley, H. Hu, E. Lafortune, and J. Bloksom. A coupled matte-specular reflection model. In *Submitted to Sigghraph 97*, 1997.

[12] P. Shirley, B. Wade, P. Hubbard, and D Zareski. Global illumination via density-estimation radiosity. In *Eurographics Rendering Workshop '95*, 1995.

[13] B. Smits and H. W. Jensen. Global illumination test scenes. Technical Report University of Utah, Technical Report UUCS-00-013, 2000. www.cs.utah.edu/~bes/papers/scenes.

[14] L. Szirmay-Kalos. *Monte-Carlo Methods in Global Illumination*. Institute of Computer Graphics, Vienna University of Technology, Vienna, 1999. http://www.iit.bme.hu/~szirmay.

[15] L. Szirmay-Kalos. Stochastic iteration for non-diffuse global illumination. *Computer Graphics Forum (Eurographics'99)*, 18(3):233–244, 1999.

[16] L. Szirmay-Kalos and W. Purgathofer. Analysis of the quasi-monte carlo integration of the rendering equation. In *Winter School of Computer Graphics '99*, pages 281–288, Plzen, Czech Republic, 1999.