CONSTRUCTION OF NON-UNIFORM BASIS FUNCTIONS FOR SPLINE CURVES AND SURFACES

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ABSTRACT

Non-uniform basis functions for construction of interpolating and approximating spline curves and surfaces are presented. The construction is based on the theory of B-splines and enables a continuous change from interpolation to approximation of given data. It is also possible to change the tension of the curves and surfaces.

Keywords: B-spline curves, geometric modelling, interpolation, approximation, basis functions, tensor-product surfaces, spline-blended surfaces.

1. INTRODUCTION

The fundamental problem of geometric modelling is the construction of smooth parametric curves and surfaces to model real objects mainly in industrial design, for example a shoe or a body of a car. Many suitable methods were developed, for example approximating B-splines, see [1].

The problem of approximation of data set was sufficiently solved in the B-spline theory. C_2-continuous cubic B-splines are probably the best for the practical use. The control points affect the shape of the curve or surface only locally, what is a common request for the practical use. Known interpolating curves are either only C_1-continuous, what means not smooth enough, or the control points affect the curve or surface globally. Other common request is a possibility of a continuous change of the approximating character of the curve to interpolating. It is not always easy to decide whether it is better to use the interpolation or the approximation. The interpolation is more natural, but highlights a possible noise. The approximation is more suitable for noisy data, see Fig. 1,2,3. And finally, it is easier to manipulate data of a curve or surface than approximated control points.

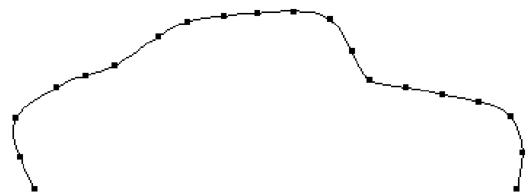

Fig. 1
Interpolating curve

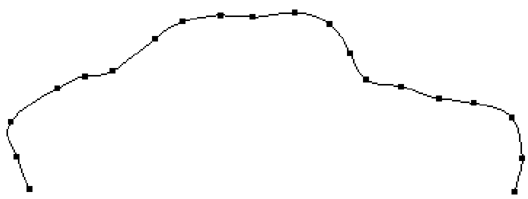

Fig. 2
Interpolating curve and noisy data

In this paper we will construct an C_2-continuous interpolating curve with local affect of control points. Then we will generalise it to a set of approximating curves containing the B-spline curve. Finally we will generalise this construction for tensor-product and spline-blended surfaces.

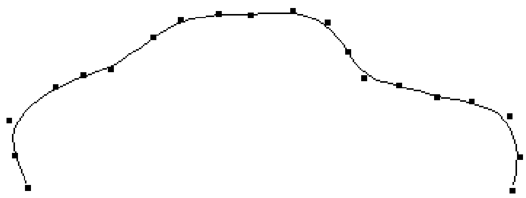

Fig. 3
Approximating curve and noisy data

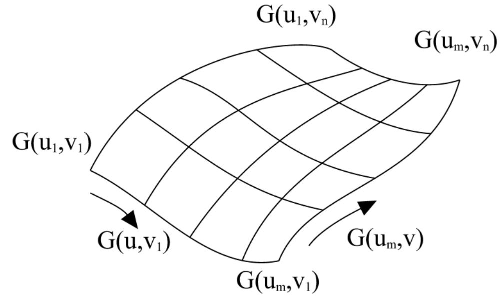

Fig. 4
Network of curves

2. SPLINE CURVE AND SURFACE CONSTRUCTION METHODS

A spline curve can be defined using control vertices and basis functions. The curve approximates the shape of the control polygon. Let us denote $W_1,\ldots,W_n$ the control vertices of the curve and $u_1,\ldots,u_n$ its knot sequence. A spline curve is defined as follows:

$$L(u)=\sum_{i=1}^{n} W_i L_i^n(u) \tag{1}$$

where $L_i^m(u)$ are the basis functions. The curve is correctly defined if the following equation is satisfied:

$$\sum_{i=1}^{n} L_i^n(u)=1,\quad \forall u\in[u_1,u_n]$$

The curve is interpolating, if

$$L_i^n(u_i)=1,\quad \mathrm{i}=1,\ldots,n. \tag{2}$$

A tensor-product surface is defined as follows:

$$S(u,v)=\sum_{i=1}^{m}\sum_{j=1}^{n} W_{ij} L_i^m(u) L_j^n(v)$$

(3)

where W_{ij}, i=1,...,m, j=1,...,n are the control vertices, $L_i^m(u)$ are the basis functions and the surface approximates the shape of the control mesh.

A spline-blended surface is a surface interpolating a given network of curves defined as follows:

$$\begin{aligned}
G(u,v)&=G_1(u,v)+G_2(u,v)-G_{12}(u,v)\\
G_1(u,v)&=\sum_{i=1}^{m} G(u_i,v) L_i^m(u)\\
G_2(u,v)&=\sum_{j=1}^{n} G(u,v_j) L_j^n(v)\\
G_{12}(u,v)&=\sum_{i=1}^{m}\sum_{j=1}^{n} G(u_i,v_j) L_i^m(u) L_j^n(v)
\end{aligned} \tag{4}$$

where $G(u,v_j)$, j=1,...,n are u-curves and $G(u_i,v)$, i=1,...,m are v-curves and $L_i^m(u)$ are the basis functions called also blending functions, see Fig. 4.

3. CONSTRUCTION OF BASIS FUNCTIONS

We will construct the basis functions as B_2-spline functions, which are similar to the B-spline functions. Instead of $2n+1$ B-spline control vertices $D_0,\ldots,D_{2n}$, a B_2-spline is determined by even control vertices $D_0,D_2,\ldots,D_{2n}$. Odd control vertices $D_1,D_3,\ldots,D_{2n-1}$ are replaced by points P_i, $i=1,\ldots,n$ joining the $(2i-1)$-th and the $2i$-th segment. Let us denote the knot sequence $u'_0,\ldots,u'_{2n+2}$. The joint P_i is given by the equation:

$$\begin{aligned}
P_i={}&\frac{\Delta'_{2i}}{\Delta'_{2i-1}+\Delta'_{2i}}\cdot\frac{\Delta'_{2i}}{\Delta'_{2i-2}+\Delta'_{2i-1}+\Delta'_{2i}}D_{2i-2}+\\
&+\left[\frac{\Delta'_{2i}}{\Delta'_{2i-1}+\Delta'_{2i}}\cdot\frac{\Delta'_{2i-2}+\Delta'_{2i-1}}{\Delta'_{2i-2}+\Delta'_{2i-1}+\Delta'_{2i}}+\right.\\
&\left.+\frac{\Delta'_{2i-1}}{\Delta'_{2i-1}+\Delta'_{2i}}\cdot\frac{\Delta'_{2i}+\Delta'_{2i+1}}{\Delta'_{2i-1}+\Delta'_{2i}+\Delta'_{2i+1}}\right]\cdot D_{2i-1}+\\
&+\frac{\Delta'_{2i-1}}{\Delta'_{2i-1}+\Delta'_{2i}}\cdot\frac{\Delta'_{2i}}{\Delta'_{2i-1}+\Delta'_{2i}+\Delta'_{2i+1}}\cdot D_{2i}
\end{aligned} \tag{5}$$

where $\Delta'_j=u'_{j+1}-u'_j$.

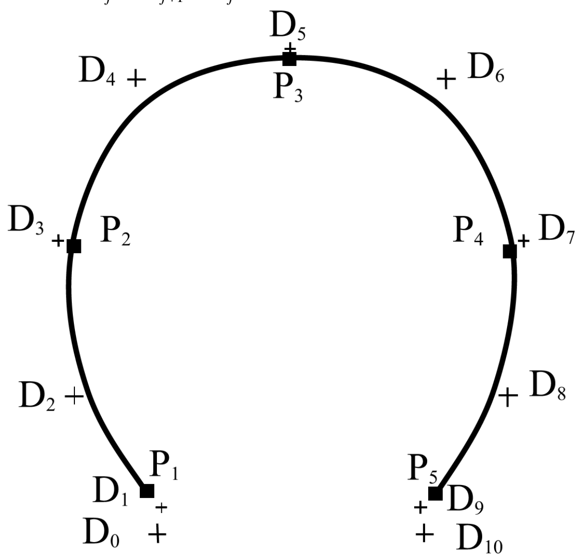

Fig. 5
B_2-spline curve

The equation for curves is the same as the equation for functions (2). To sum up, B_2-spline function or curve interpolates its joints. Even control vertices affect the shape of the function or curve, see Fig 5. The following equation for enumeration of B_2-spline functions (or curves) can be derived:

$$Q_{2i-1}(t) = (t^3, t^2, t, 1)\frac{1}{24}\begin{pmatrix} -7 & 18 & -16 & 6 & -1 \\ 18 & -36 & 18 & 0 & 0 \\ -12 & 0 & 12 & 0 & 0 \\ 0 & 24 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} D_{2i-2} \\ P_i \\ D_{2i} \\ P_{i+1} \\ D_{2i+2} \end{pmatrix}$$

$$Q_{2i}(t) = (t^3, t^2, t, 1)\frac{1}{24}\begin{pmatrix} 1 & -6 & 16 & -18 & 7 \\ -3 & 18 & -30 & 18 & -3 \\ 3 & -18 & 0 & 18 & -3 \\ -1 & 6 & 14 & 6 & -1 \end{pmatrix}\begin{pmatrix} D_{2i-2} \\ P_i \\ D_{2i} \\ P_{i+1} \\ D_{2i+2} \end{pmatrix}$$

where $t=u-(2i-1)$ or $t=u-2i$.

In [3] a spline curve has been constructed, and it has been determined by control vertices $W_1,...,W_n$ and B_2-spline basis functions, defined by equation (1). In [2] a relationship between B_2-spline control vertices of basis functions and B_2-spline control vertices of a spline curve has been derived. Let P_i^j and D_{2i}^j be joints and even control vertices of j-th basis function, respectively. Let C_{2k} and R_k be even control vertices and joints of B_2-spline curve, respectively. These vertices satisfy the equation:

$$R_k = \sum_{i=1}^n W_i P_k^i \qquad C_{2k} = \sum_{i=1}^n W_i D_{2k}^i \qquad (6)$$

In this equation the control vertices of the basis functions $D_0,D_2,\dots,D_{2n}$ and $P_1,\dots,P_n$ are considered to be only real numbers equal to the second coordinate of the points.

In [3] we have derived B_2-spline control vertices for the uniform interpolating and approximating curve with control vertices $W_1,...,W_n$:

$$R_i = a \cdot W_{i-1} + (1-2a) \cdot W_i + a \cdot W_{i+1}$$

$$C_{2i} = -p \cdot W_{i-1} + (\tfrac{1}{2} + p) \cdot W_i + (\tfrac{1}{2} + p) \cdot W_{i+1} - p \cdot W_{i+2}$$

where a is a parameter of approximation and p is a parameter of the tension. For $a=0$ we get an interpolating curve and for $a=1/6$ and $p=0$ we get the uniform B-spline curve.

This construction gives good results only for approximately equidistant data points, see Fig. 6,7,8.

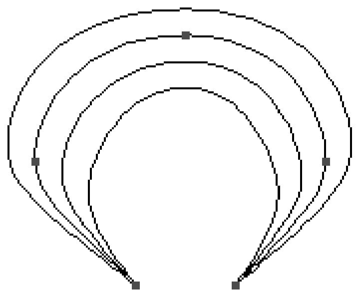

Fig. 6
Parameter of approximation a=-0.1, 0, 0.1, 0.2

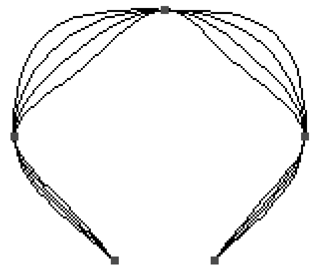

Fig. 7
Parameter of tension p=0, 0.1, 0.2, 0.3

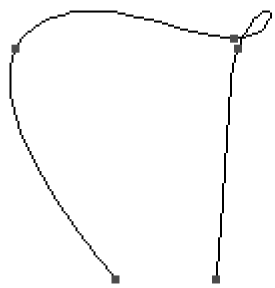

Fig. 8
Problem of the uniform curve

4. NON-UNIFORM INTERPOLATING CURVE

In this section we will solve a problem shown in Fig. 8 and then generalise the uniform interpolating curve constructed in Section 3 to non-uniform.

Using a non-uniform knot sequence in the theory of B-spline curves can solve the problem shown in Fig. 8. Let $C'_{-1},...,C'_{L+2}$ be control points and $u_0,...,u_{L+4}$ a knot sequence of cubic B-spline curve consisting of L segments. If the knot sequence is determined by the equation

$$u_{i+1} = u_i + \operatorname{dist}\left(C'_{i-1}C'_i\right), \quad i = 2,..., L+3$$

the shape of curve is satisfactory. This method is called chord-length parameterisation. We will try to solve the problem of the interpolating curve using chord-length parameterisation, too.

Let us have a B_2-spline curve interpolating its joints $W_1,\dots,W_n$, with even control vertices $C_0,C_2,\dots,C_{2n}$ and a knot sequence $u'_0,\dots,u'_{2n+2}$ determined as follows:

$$u'_0 = u'_1 = u'_2 = 0, \; u'_{2n+2} = u'_{2n+1} = u'_{2n}$$
$$u'_{2i+2} = u'_{2i} + \operatorname{dist}(W_{i-1}\, W_i), \; i=2,\dots,n-1$$
$$u'_{2i+1} = (u'_{2i} + u'_{2i+2})/2, \; i=2,\dots,n$$

We can also get the odd B-spline vertices using this knot sequence and the equation (5). Then the known DeBoore algorithm can be used to enumerate the points of the curve.

Now it is necessary to find appropriate even control vertices $C_0,C_2,\dots,C_{2n}$. We will find even control vertices of a symmetrical B-spline curve consisting of 3 segments and use this result to find suitable even control vertices of the B_2-spline curve.

Let us consider a B-spline curve (see Fig. 9) consisting of 3 segments, interpolating joints W_1 = (0,0), W_2=(0,C), W_3=(B,C), W_4= (B,0) with knot sequence $u_0,\ldots,u_7$, where u_0=0, $\Delta_1=A$, $\Delta_2=C$, $\Delta_3=B$, $\Delta_4=C$, $\Delta_5=A$, ($\Delta_i=u_{i+1}-u_i$), u_1 and u_7 will be chosen later. Now it is possible to find the B-spline control points $C'_0,\ldots,C'_5$ of the curve, where C'_0,C'_5, u_1 and u_7 are chosen so that $W_1=C'_1$ and $W_4=C'_4$.

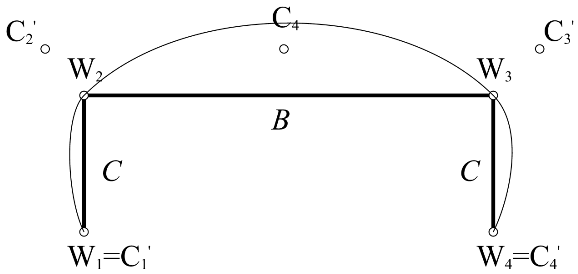

Fig. 9

Now we will split each segment defined on $[u_i, u_{i+1})$ in the knot $(u_i+u_{i+1})/2$, find its new B-spline control vertices $C_0,\ldots,C_8$ and find a relationship between its joints W_1,W_2,W_3,W_4 and even control vertices $C_0, C_2,\ldots,C_8$ (and the parameter of tension p).

It is possible to prove, that if $C'_0,\ldots,C'_{n+1}$ are B-spline control points of the original curve and $C_0,\ldots,C_{2n}$ are B-spline control points of the curve with split segments, the control vertices satisfy these equations:

$$C_{2i} = \frac{\Delta_i/2+\Delta_{i+1}}{\Delta_{i-1}+\Delta_i+\Delta_{i+1}}\cdot C'_i + \frac{\Delta_{i-1}+\Delta_i/2}{\Delta_{i-1}+\Delta_i+\Delta_{i+1}}\cdot C'_{i+1}$$

$$C_{2i-1} = x_1^i\cdot C'_{i-1} + x_2^i\cdot C'_i + x_3^i\cdot C'_{i+1}$$

$$x_1^i = \frac{1}{4}\cdot\frac{\Delta_{i-1}+2\Delta_i}{\Delta_{i-2}+\Delta_{i-1}+\Delta_i}\cdot\frac{\Delta_i}{\Delta_{i-1}+\Delta_i}$$

$$x_3^i = \frac{1}{4}\cdot\frac{2\Delta_{i-1}+\Delta_i}{\Delta_{i-1}+\Delta_i+\Delta_{i+1}}\cdot\frac{\Delta_{i-1}}{\Delta_{i-1}+\Delta_i} \qquad (7)$$

$$x_2^i = 1 - x_1^i - x_3^i$$

We will investigate the y-coordinate of the point $C_4=(B/2,y)$. This point satisfies the equation

$$C_4 = -p\cdot W_1 + (\tfrac{1}{2}+p)\cdot W_2 + (\tfrac{1}{2}+p)\cdot W_3 - p\cdot W_4$$

If we consider only the y-coordinate of this equation we get $y=(1+2p)C$, and we get the equation $p=(y/C-1)/2$. Using the equations (7) a (5) we get

$$p = \frac{B^2}{A(B+C)+2BC+C^2}$$

Now we can choose a value of A. Let us suppose B=C. Let us choose $\frac{5}{4}B$ to be a value of the y-coordinate of the control vertex C'_4. This condition gives $A=\frac{1}{2}B$ and

$$p = \frac{B^2}{B^2+5BC+2C^2}$$

This is a symmetrical curve. Now we can generalise the result for a non-symmetrical curve:

$$C_{2i} = -p_i^-\cdot W_{i-1} + (\tfrac{1}{2}+p_i^-)\cdot W_i + (\tfrac{1}{2}+p_i^+)\cdot W_{i+1} - p_i^+\cdot W_{i+2} \qquad (8)$$

$$p_i^- = \frac{\Delta_i^2}{\Delta_i^2+5\Delta_i\Delta_{i-1}+\Delta_{i-1}^2} \quad p_i^+ = \frac{\Delta_i^2}{\Delta_i^2+5\Delta_i\Delta_{i+1}+\Delta_{i+1}^2}$$

We have constructed an interpolating curve not causing problems shown in Fig. 8 as it can be seen in Fig. 10.

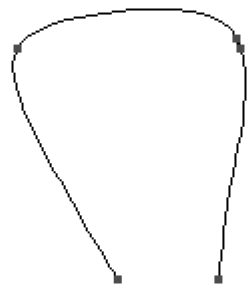

Fig. 10
Problem of Fig. 8 is solved

The curve is suitable to model objects determined by few points. But if we are given a lot of data for a simple object, the curve does not seem to be very suitable, see Fig. 11.

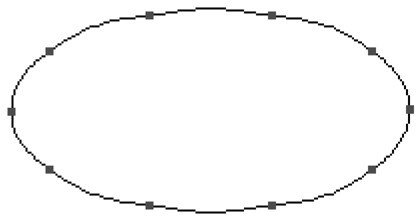

Fig. 11

So we will consider the parameter of tension p again:

$$p_i^- = \frac{6\cdot p\cdot\Delta_i^2}{\Delta_i^2+5\Delta_i\Delta_{i-1}+\Delta_{i-1}^2} \quad p_i^+ = \frac{6\cdot p\cdot\Delta_i^2}{\Delta_i^2+5\Delta_i\Delta_{i+1}+\Delta_{i+1}^2}$$

If we use a tension p=0.13 for the object in Fig. 11, we get a good result (Fig. 12).

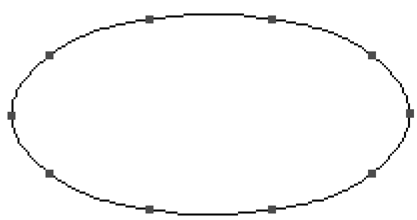

Fig. 12

But sometimes we need different values of the tension in different parts of the object, so we define the parameter of tension p_i for each segment separately:

$$p_i^- = \frac{6 \cdot p_i \cdot \Delta_i^2}{\Delta_i^2 + 5\Delta_i\Delta_{i-1} + \Delta_{i-1}^2} \qquad p_i^+ = \frac{6 \cdot p_i \cdot \Delta_i^2}{\Delta_i^2 + 5\Delta_i\Delta_{i+1} + \Delta_{i+1}^2} \tag{9}$$

Let us summarise the definition of the interpolating curve. Given:

- control points $W_{-1},\dots,W_{n+2}$
- a knot sequence $u_{-1},\dots,u_{n+2}$
- a parameter of tension sequence $p_1,\dots,p_{n-1}$

Interpolating non-uniform curve defined on $[u_1,u_n]$, consisting of n-1 segments, interpolates control points $W_1,\dots,W_n$, the i-th segment is defined on $[u_i,u_{i+1})$ and parameter of tension p_i belongs to this segment. The curve is the same as B-spline curve consisting of $2n$-2 segments determined by control vertices $C_0,\dots,C_{2n}$ satisfying the equations (5),(8),(9) and the knot sequence $u'_0,\dots,u'_{2n+2}$ is defined as follows:

$u'_{2i}= u_i$, $i=0,\dots,n+1$; $u'_{2i+1}=(u_i +u_{i+1})/2$, $i=0,\dots,n$;

We can find one more problem in this construction for a special kind of the control polygon, see Fig. 13. The curve does not approximate this control polygon very accurately.

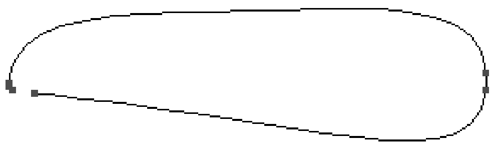

Fig. 13

The solution is in a little modification of the knot sequence:

$u'_{2i}=(u_{i-1}+K u_i+ u_{i+1})/(K+2)$, $i=0,\dots,n+1$;

$u'_{2i+1}=(u_i +u_{i+1})/2$, $i=0,\dots,n$;

where K is big enough (for example K=20).

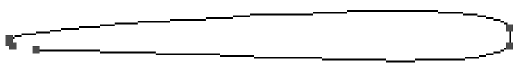

Fig. 14

Problem of Fig. 13 is solved

To finish this section, we suggest a way how to determine suitable values of parameters of tension $p_1,\dots,p_{n-1}$. Our equation has been constructed to satisfy these conditions:

- If the direction of the control polygon edges is changed only a little, the tension is close to zero, see Fig. 15.

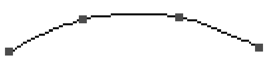

Fig. 15

- If the direction is changed quickly, the tension rises, see Fig. 16.

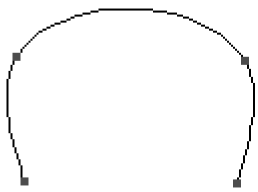

Fig. 16

- If the direction is changed almost opposite way, the tension is close to zero again.

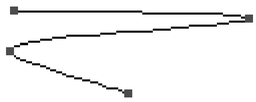

Fig. 17

- If the direction of an edge is changed and the next edge is returned to the original direction, the tension is again close to zero.

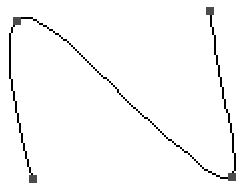

Fig. 18

We have suggested the following equation:

$$p_i = p \cdot \frac{|\cos\alpha_i + \cos\alpha_{i+1}|}{2} \qquad \alpha_i = \angle W_{i-1}W_iW_{i+1}$$

where p is the parameter of tension. Suitable default value is p=0.2. Fig. 19 shows that this method is much better in detail (compare Fig. 1).

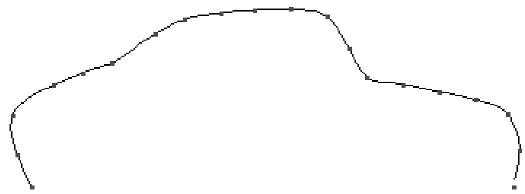

Fig. 19

Interpolating curve, p=0.2

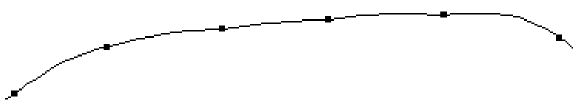

Fig. 20

Detail of Fig. 1

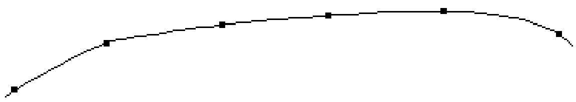

Fig. 21

Detail of Fig. 19

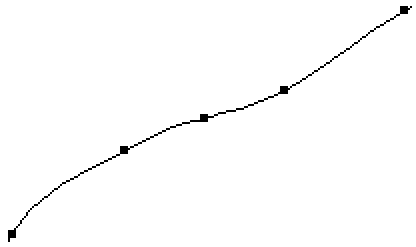

Fig. 22
Detail of Fig. 1

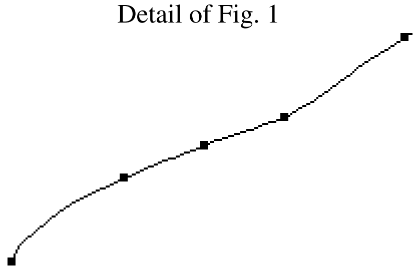

Fig. 23
Detail of Fig. 19

5. APPROXIMATING CURVE

In this section we will generalise the interpolating curve to a system of approximating curves with the parameter of approximation a.

For a=0 we get the interpolating curve and for a=1/6 and p_i=0, i=1,…,n–1 it is supposed to be the B-spline curve. The approximating curves for other values of parameter a will be constructed as linear interpolation of the interpolating and the B-spline curve. Advantage of this construction is that we can interpolate the control points of the interpolating and the B-spline curve and get the same result.

Let us have a B-spline curve $L^B(u)$ with B-spline control vertices $W_0,\dots,W_{n+1}$ and a knot sequence $u_{-1},\dots,u_{n+2}$ (the B-spline curve does not depend on vertices W_{-1} and W_{n+2}). Let us denote its B_2-spline control vertices: the joints are $W_1^B,\dots,W_n^B$ and the even control vertices are $C_0^B, C_2^B,\dots,C_{2n}^B$. Its parameter of tension sequence is $p_i^B = 0,\ i = 1,\dots,n-1$. We can get these vertices from given points $W_{-1},\dots,W_{n+2}$ using the equations (7) and (5). The control points of the interpolating curve $L(u)$ are denoted in the same way as in previous Section: $W_{-1},\dots,W_{n+2}$ are the joints, $C_0,\dots,C_{2n}$ are the even control vertices and $p_1,\dots,p_{n-1}$ is the parameter of tension sequence.

Let us denote $L^a(u)$ an approximating curve with parameter of approximation a. Let us have $L^0(u) = L(u)$ and $L^{1/6}(u) = L^B(u)$. $L^a(u)$ is defined as follows:

$$L^a(u) = 6\cdot a\cdot L^B(u) + (1-6\cdot a)\cdot L(u)$$

The B_2-spline control vertices of the approximating curve $L^a(u)$ satisfy:

$$W_i^a = 6\cdot a\cdot W_i^B + (1-6\cdot a)\cdot W_i,\ i=1,\dots,n$$

$$C_{2i}^a = 6\cdot a\cdot C_{2i}^B + (1-6\cdot a)\cdot C_{2i},\ i=0,\dots,n$$

6. TENSOR-PRODUCT SURFACES

It is quite easy to generalise the definition of the approximating curve for tensor-product surfaces:
Given:

- control points W_{ij}, i= -1,…, m+2, j= -1,…,n+2
- knot sequences $u_{-1},\dots,u_{m+2}$, $v_{-1},\dots,v_{n+2}$
- parameter of tension sequences $p_1,\dots,p_{m-1}$, $q_1,\dots,q_{n-1}$

The curve defined in previous Section is denoted as $C(v)$ and the tensor-product surface is denoted as $S(u,v)$.
According to the equations (1) and (3) we can write

$$C(v) = \sum_{i=1}^{n} W_j L_j^n(v)$$

$$S(u,v) = \sum_{i=1}^{m}\left[\sum_{j=1}^{n} W_{ij} L_j^n(v)\right] L_i^m(u) = \sum_{i=1}^{m} C^i(v) L_i^m(u)$$

where $C^i(v)$ is the approximating curve defined in the previous Section by the control vertices $W_{i,-1},\dots,W_{i,n+2}$ and the knot sequence $v_{-1},\dots,v_{n+2}$. We can also write $S(u,v)=C^v(u)$, where $C^v(u)$ is the approximating curve defined in the previous Section by the control vertices $C^{-1}(v),\dots, C^{m+2}(v)$ and the knot sequence $u_{-1},\dots,u_{m+2}$.

It is not possible to generalise suggestion of the knot sequences and the parameter of tension sequences because we have to suggest only one knot sequence and parameter of tension sequence for all u-curves and a second for v-curves. This is the reason why we do not give any suggestions for tensor product surfaces. Both knot sequences and parameter of tension sequences must be suggested separately for specific needs of each application.

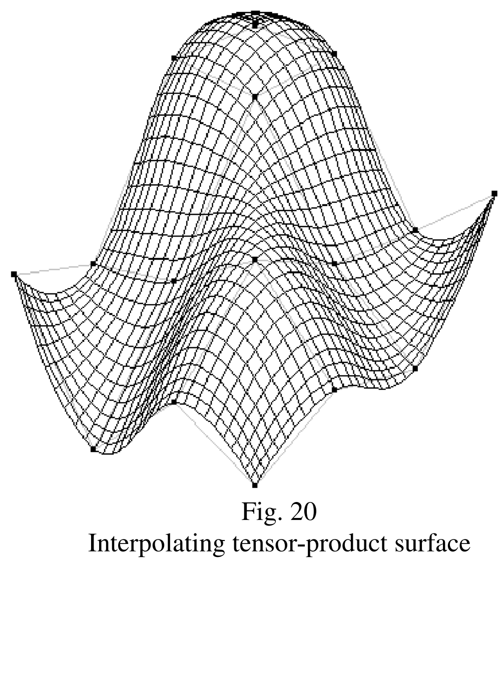

Fig. 20
Interpolating tensor-product surface

7. SPLINE-BLENDED SURFACE

Given:

- knot sequences $u_{-1},\ldots,u_{m+2}$, $v_{-1},\ldots,v_{n+2}$
- a network of curves $G(u_i,v)$, i= -1,…, m+2, $G(u,v_j)$, j= -1,…,n+2
- parameter of tension sequences $p_1,\ldots,p_{m-1},$ $q_1,\ldots,q_{n-1}$

The points of the spline-blended surface defined in (4) can be enumerated in the similar way as in previous Section. The surface G_{12} from definition (4) is a tensor-product surface. If we fix the parameter v in the surface G_1 and the parameter u in the surface G_2, we get the definition of a curve according to (1). So we can use algorithm for curves.

8. CONCLUSION

We have constructed an appropriate non-uniform system of spline curves containing an interpolating curves and the B-spline curve. The interpolating curve is special case of the B_2-spline curve, but we have found appropriate even control points. User does not have to take care of them, but they can also be changed using the parameter of tension sequence. The results of this paper have been implemented and we represent them in several pictures shown in this paper. We think it would be useful to generalise these results to a system of non-uniform rational curves, containing the most used NURBS and try other parameterisation methods as well.

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