BEZIER CURVES INTERSECTION USING RELIEF PERSPECTIVE

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ABSTRACT

Presented paper describes the method for finding the intersection of class space rational Bezier curves. The problem curve/curve intersection belongs among basic geometric problems and the aim of this article is to describe the new technique to solve the problem using relief perspective and Bezier clipping.

Keywords: Bezier curve, perspective collineation, Bezier clipping

1 INTRODUCTION

Curve/curve intersection is one of the fundamental problems of computational geometry. At the present time there exist several different approaches to this problem but the endeavor is to avoid difficulties in calculations which are mainly results of polynomial representation rational higher degree curves. It can be done in certain cases by using the relief perspective and Bezier clipping.

Bezier clipping, in the context of plane curve intersection in this paper, see e.g. [Nishi90] [Nishi98], is an interactive method which takes advantages of the convex hull property of Bezier curves. Regions of one curve which are guaranteed do not intersect a second curve can be identified and subdivided away.

Relief perspective is a mapping of space into space in order to correspond conditions of the human seeing. Images of all objects under the relief perspective are located in space between two parallel planes.

2 RELIEF PERSPECTIVE

The relief perspective is a special case of perspective collineation of the extended Euclidean space $\bar{\mathbb{E}}_3$. We remind that a collineation is a bijective mapping $\varphi : \bar{\mathbb{E}}_3 \to \bar{\mathbb{E}}_3$ that preserves collinearity of points. If there exists a plane ω such that $\varphi(X) = X \ \forall X \in \omega$, φ is a perspective collineation (for more details, see e.g. [Bus53] [Cizm84]). Then there exists a point O (called the centre of perspective collineation) such that $\varphi(O) = O$ and $\varphi(\alpha) = \alpha \ \forall\alpha;\ O \in \alpha$.

The relief perspective is a perspective collineation including some additional conditions:

- in order to produce correct images of 3D objects, it is important to respect necessary conditions of the human seeing. As in linear perspective, see e.g. [Cenek59], also in the relief perspective we suppose, that objects are located inside viewing circular cone. The cone has a vertex in the eye (the centre of projection) and the distance from the eye to the objects has to be at least 25 cm.
- the image plane ω (the set of invariant points) does not contain the centre O (i.e. φ is a homology) and determine elements of the mapping φ are: the centre O, the image plane σ (the set of invariant points) and the vanishing plane ω^r (the image of the plane at infinity).
- no objects are ideal, i.e. objects and their

images in the relief perspective have not ideal points. In notions of $\mathbb{E}_3$ the image plane has to be placed between the point O and the plane ω^r. The mapped object, that image we want to construct, is located in the semi-space, that it is opposite to the semi-space $\overrightarrow{\sigma O}$ ("behind the image plane σ").

Let φ be a relief perspective of space $\bar{\mathbb{E}}_3$. Let denote $\bar{\mathbb{E}}_3$ be a preimage space and $\bar{\mathbb{E}}_3^r$ be an image space ($\bar{\mathbb{E}}_3$ with upper index "r") such that $\varphi : \bar{\mathbb{E}}_3 \to \bar{\mathbb{E}}_3^r$. The image of the object $\mathcal{U} \in \bar{\mathbb{E}}_3$ under the relief perspective φ is called the relief of the object $\mathcal{U}$ and is denoted analogically $\mathcal{U}^r$. The relief perspective is given by the centre O, the image plane σ and ordered pair of different points $A, A^r = \varphi(A)$ $(A, \varphi(A) \neq O)$ such that $A, A^r \notin \sigma$ and O, A, A^r are collinear.

Note therefore all mapped objects $\mathcal{U}$ and their images $\mathcal{U}^r$ under the relief perspective have no ideal points, we can study them in $\mathbb{E}_3$ or $\bar{\mathbb{E}}_3$ using affine and projective methods as well. It is also clear that the restriction $\varphi/\mathcal{U}$ of the relief perspective φ, which is a bijective mapping, is again a bijection.

3 REPRESENTATION OF THE RELIEF PERSPECTIVE IN AN ANALYTICAL FORM

Suppose X and X' are two different points of $\mathbb{E}_3 \subseteq \bar{\mathbb{E}}_3$ with coordinates $[x, y, z]$ and $[x', y', z']$ respectively. If φ is a collineation of the space $\bar{\mathbb{E}}_3$ we use the following equations:

$$x' = \frac{a_{11}x + a_{12}y + a_{13}z + a_{14}}{a_{41}x + a_{42}y + a_{43}z + a_{44}}$$
$$y' = \frac{a_{21}x + a_{22}y + a_{23}z + a_{24}}{a_{41}x + a_{42}y + a_{43}z + a_{44}} \qquad (1)$$
$$z' = \frac{a_{31}x + a_{32}y + a_{33}z + a_{34}}{a_{41}x + a_{42}y + a_{43}z + a_{44}},$$

where $a_{i,j}$ $(i, j \in \{1..4\})$ are real numbers and $det(a_{ij}) \neq 0$. In equations (1) are used non-homogenic coordinates, which correspond to a restriction $\bar{\varphi} = \varphi/\{\bar{\mathbb{E}}_3 - \alpha\}$, where α is a plane $a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_0 = 0$.

In order to obtain simple mapping equations for relief perspective from (1) we assume, that the centre O is a point with coordinates $[0, 0, 0]$ and σ is a plane $z - 1 = 0$. In this special case we obtain the convenient representation of the relief perspective in the following form

$$x' = \frac{(1+k)x}{z+k}$$
$$y' = \frac{(1+k)y}{z+k} \qquad (2)$$
$$z' = \frac{(1+k)z}{z+k}$$

where the span $k \in \mathbb{R}^+ - \{1\}$ (the geometric representation of the parameter k will be explained in the next lines).

4 SOME PROPERTIES OF THE RELIEF PERSPECTIVE

- the vanishing plane $\omega^r \equiv 1 + k$ and the neutral plane $\nu \equiv -k$ (the preimage of the plane at infinity) are parallel and for distances O, ν and ω^r, σ we have

$$| O\nu | = | \omega^r \sigma | = k$$

  where the parameter k represents the span

- if a plane $\alpha \equiv z - c$ $(c \neq 0)$ is parallel to the image plane $(\alpha \neq \sigma, \nu)$, then the relief of the plane α is the plane $\alpha^r \equiv \frac{(1+k)c}{c+k}$ and these planes are parallel (it is obvious that $\alpha \parallel \alpha^r \parallel \sigma$)

- the relief of a point placed in the semi-space, that is opposite to the semi-space $\overrightarrow{\sigma O}$, is a point in a part of space with boundary planes $z = 1$ and $z = 1 + k$ (in an intersection of spaces $z > 1$ and $z < 1 + k$) (Fig. 1)

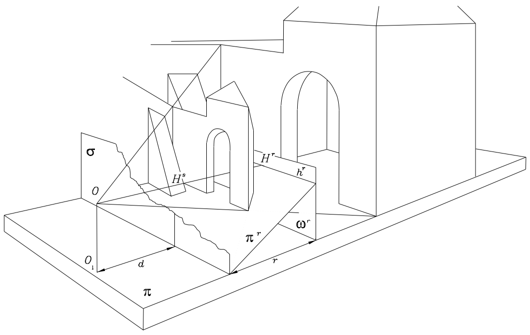

Figure 1

- the orthographic projection of the object $\mathcal{U}^r$ (the relief of the object $\mathcal{U}$) onto the image plane σ is the central projection of the object $\mathcal{U}$ from the point S, where $S \in l \cap \nu$ $\quad (l; O \in l \wedge l \perp \sigma)$

5 RATIONAL BEZIER CURVES AND RELIEF PERSPECTIVE

The relief perspective is a mapping of the extended Euclidean space $\bar{\mathbb{E}}_3$. Let V_i, $i \in \{0,1,\ldots,n\}$ be given points, $[x_i, y_i]$ be their coordinates of the space $\bar{\mathbb{E}}_2$ and positive real numbers w_i, $i \in \{0,1,\ldots,n\}$ be their weights. By these elements and Bernstein polynomials $B_i^n(t)$, see e.g. [Farin93], is defined a n-th planar rational Bezier curve

$$\mathbf{P}(t) = [\frac{\sum_{i=0}^{n} x_i w_i B_i^n(t)}{\sum_{i=0}^{n} w_i B_i^n(t)}, \frac{\sum_{i=0}^{n} y_i w_i B_i^n(t)}{\sum_{i=0}^{n} w_i B_i^n(t)}] \qquad (3)$$
$$t \in \langle 0,1\rangle$$

and also the space nonrational Bezier curve

$$\bar{\mathbf{P}}(t) = [\sum_{i=0}^{n} x_i w_i B_i^n(t), \sum_{i=0}^{n} y_i w_i B_i^n(t), \sum_{i=0}^{n} w_i B_i^n(t)], \quad t \in \langle 0,1\rangle \qquad (4)$$

with control points

$$V_i = [x_i w_i,\, y_i w_i,\, w_i]. \qquad (5)$$

Let $\varphi : \bar{\mathbb{E}}_3 \to \bar{\mathbb{E}}_3^r$ be the relief perspective of the space $\bar{\mathbb{E}}_3$ and $X^r[x,y,z]$ ($X^r \notin \sigma$) be the relief of the point $X[a,b,c] \in \bar{\mathbb{E}}_3$. From the equations (2) we get

$$x = \frac{(1+k)a}{c+k},\ y = \frac{(1+k)b}{c+k},\ z = \frac{(1+k)c}{c+k} \qquad (6)$$

and according to the relation $z = \frac{(1+k)c}{c+k}$ we obtain

$$c + k = \frac{k(1+k)}{1+k-z}$$

From the above results coordinates $[a,b,c]$ of the point X, that is the preimage of the relief $X^r[x,y,z]$, are computed according to $a = \frac{kx}{1+k-z}$, $b = \frac{ky}{1+k-z}$, $c = \frac{kz}{1+k-z}$.

Lemma *Let $\varphi : \bar{\mathbb{E}}_3 \to \bar{\mathbb{E}}_3^r$ be relief perspective of the space $\bar{\mathbb{E}}_3$. The point $[x,y,z]$ is the relief of the point $[\frac{kx}{1+k-z}, \frac{ky}{1+k-z}, \frac{kz}{1+k-z}]$.*

By given planar points V_i and their weights w_i, where $1 + k > w_i\ \forall i$, we can define in $\bar{\mathbb{E}}_3$ points

$$W_i = [\frac{kx_i w_i}{1+k-w_i}, \frac{ky_i w_i}{1+k-w_i}, \frac{kw_i}{1+k-w_i}]$$
$$i \in \{0,1,\ldots,n\} \qquad (7)$$

With respect to the lemma above it is obvious, that the reliefs of the points W_i are represented by the points (5). Using the points V_i, the space nonrational Bezier curve $\bar{\mathbf{P}}(t)$ of the form (4) is defined and we notice, that $\bar{\mathbf{P}}(t)$ under mapping φ is an image of the curve

$$\mathbf{Q}(t) = [\frac{\sum_{i=0}^{n} kx_i w_i B_i^n(t)}{\sum_{i=0}^{n}(1+k-w_i)B_i^n(t)}, \frac{\sum_{i=0}^{n} ky_i w_i B_i^n(t)}{\sum_{i=0}^{n}(1+k-w_i)B_i^n(t)}, \frac{\sum_{i=0}^{n} kw_i B_i^n(t)}{\sum_{i=0}^{n}(1+k-w_i)B_i^n(t)}] \qquad (8)$$
$$t \in \langle 0,1\rangle.$$

This curve $\mathbf{Q}$(t) is the space rational Bezier curve defined by control points of the form (7) and their weights Ω_i, that are computed according to $\Omega_i = 1 + k - w_i$ because of
$\frac{kx_i w_i}{1+k-w_i}\Omega_i = kx_i w_i$, $\quad \frac{ky_i w_i}{1+k-w_i}\Omega_i = ky_i w_i$,
and $\quad \frac{kw_i}{1+k-w_i}\Omega_i = kw_i$
Applying all these results we obtain the following theorem

Theorem *The curve $\bar{\mathbf{P}}(t)$ is a relief of the curve $\mathbf{Q}(t)$.*

In order to get more information about Bezier curves and the relief perspective, let us assume having a central projection with the centre $O[0,0,0]$ and the plane $z - 1 = 0$. In this case the planar rational curve $\mathbf{P}(t)$ of the form (3) is the image of the space rational Bezier curve $\mathbf{Q}(t)$ defined by (8).

Let the space nonrational Bezier curve $\bar{\mathbf{P}}(t)$ expressed by (4) be given. What is the relief of this curve? We know $z + k = \sum_{i=0}^{n}(w_i + k)B_i^n(t)$ and now the relief $\bar{\mathbf{P}}^{\mathbf{r}}(t)$ of the given curve $\mathbf{P}^{\mathbf{r}}(t)$ can be written as

$$\bar{\mathbf{P}}^{\mathbf{r}}(t) = [\frac{\sum_{i=0}^{n}(1+k)x_i w_i B_i^n(t)}{\sum_{i=0}^{n}(w_i+k)B_i^n(t)}, \frac{\sum_{i=0}^{n}(1+k)y_i w_i B_i^n(t)}{\sum_{i=0}^{n}(w_i+k)B_i^n(t)}, \frac{\sum_{i=0}^{n}(1+k)w_i B_i^n(t)}{\sum_{i=0}^{n}(w_i+k)B_i^n(t)}] \qquad (9)$$

$t \in \langle 0,1\rangle$, where control vertices are points expressed as follow

$$\bar{V}_i^r = [\frac{(1+k)x_i w_i}{w_i+k}, \frac{(1+k)y_i w_i}{w_i+k}, \frac{(1+k)w_i}{w_i+k}]$$

and weights $\bar{\Omega}_i$ are computed by $\bar{\Omega}_i = w_i + k$.

6 INTERSECTION OF SPACE RATIONAL BEZIER CURVES

Let $\mathbf{P}(t)$, $\mathbf{Q}(u)$ be space rational Bezier curves of the form (8). The aim is to find their intersection.

The solution to this problem can be found using already known facts. According to the theorem, which was formed in the previous section, reliefs of space rational Bezier curves of the form (8) are space nonrational Bezier curves of the form (4) (in Fig. 2 Bezier curves of the form (8) and their reliefs for $k = 2$ are shown). The central projections of these nonrational curves from the point $[0, 0, 0]$ onto the plane $z-1 = 0$ are planar rational Bezier curves defined by (3). Their intersection is possible to find specifically by using the method of Bezier clipping. This method specifies values of the parameter $t \in \langle 0, 1\rangle$ of the curve $\mathbf{P}^{\mathbf{s}}(t)$, respectively the parameter $u \in \langle 0, 1\rangle$ of the curve $\mathbf{Q}^{\mathbf{s}}(u)$, that correspond to the common point $\mathbf{R}^{\mathbf{s}}$ (the upper indexes "r" and "s" denote the reliefs and central projections of curves or points). This point is the intersection of both curves . The preimage of the point $\mathbf{R}^{\mathbf{s}}$ in the central projection is the space point, which is the relief of the intersection of the space rational Bezier curves $\mathbf{P}(t)$ and $\mathbf{Q}(u)$.

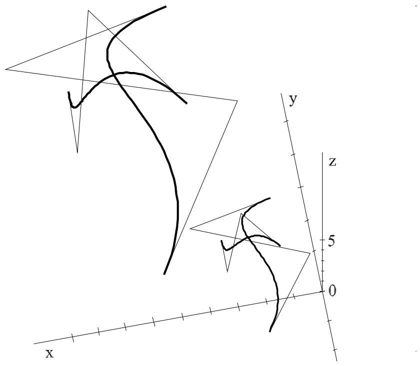

Figure 2

The following scheme shows a whole proces of finding the intersection of the $\mathbf{P}(t)$ and $\mathbf{Q}(u)$ curves:

$$\left.\begin{array}{l}\mathbf{P}(t) \xrightarrow{\varphi} \mathbf{P}^{\mathbf{r}}(t) \xrightarrow{\psi} \mathbf{P}^{\mathbf{s}}(t) \\ \mathbf{Q}(u) \xrightarrow{\varphi} \mathbf{Q}^{\mathbf{r}}(u) \xrightarrow{\psi} \mathbf{Q}^{\mathbf{s}}(u)\end{array}\right\} \xrightarrow{\text{B. clipping}} \mathbf{R}^{\mathbf{s}}$$

$$\mathbf{R}^{\mathbf{s}} = \mathbf{P}^{\mathbf{s}}(t) \cap \mathbf{Q}^{\mathbf{s}}(u) \xrightarrow{\psi^{-1}} \mathbf{R}^{\mathbf{r}} \xrightarrow{\varphi^{-1}} \mathbf{R}$$
$$\mathbf{R} = \mathbf{P}(t) \cap \mathbf{Q}(u)$$

φ – relief perspektive $\varphi : \bar{\mathbb{E}}^3 \to \bar{\mathbb{E}}^3$
ψ – central projection $\psi : \bar{\mathbb{E}}^3 \to \bar{\mathbb{E}}^2$.

The central projection ψ is not a bijective mapping and in case that the intersection of curves $\mathbf{P}^{\mathbf{s}}(t)$ and $\mathbf{Q}^{\mathbf{s}}(u)$ exists (in opposite case the given curves $\mathbf{P}(t)$ and $\mathbf{Q}(u)$ certainly do not intersect) the intersection of curves $\mathbf{P}(t)$ and $\mathbf{Q}(u)$ does not have to exist. This situation occurs when preimages of point $\mathbf{R}^{\mathbf{s}}$ on the curves $\mathbf{P}^{\mathbf{r}}(t)$ and $\mathbf{Q}^{\mathbf{r}}(u)$ in the central projection are different.

7 CONCLUSIONS AND FUTURE WORK

The relations between Bezier curves and the relief perspective have been described. The necessary and sufficient conditions for expressing the space nonrational Bezier curve as the relief of the space rational Bezier curve have been formed. We have shown that to the planar rational curve of the form (3), defined by planar points $V_i \in \bar{\mathbb{E}}_2$ and their weights w_i, it is possible to assign a class of the space curves by the relief perspective. One of them is nonrational curve defined by (4) and two are rational curves defined by (8) and (9).

The described method can be used as a direction for application how to find the intersection of the space rational curves of the form (8). It is possible to express every polynomial curve in Bezier's representation and due to this representation the method can be applied to all polynomial curves after modifications (e.g. spline curves which are considered as curves consist of Bezier segments) for solution to the curve/curve or curve/line intersection problems.

Despite the author's attempt he did not succeed in finding similar comparable published methods on website. In addition to this he is not able to compare his results with any other. Obviously author's knowledge is limited and he would appreciate to get information about any other similar method.

In the future work we want to extend possibilities of the relief perspective in geometric modeling. Our aim is to find an answer whether any polynomial 3D curve can be converted to the space curve of the form (8).

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