# Polynomiography for Square Systems of Equations with Mann and Ishikawa Iterations


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## ABSTRACT

In this paper we propose to replace the standard Picard iteration in the Newton–Raphson method by Mann and Ishikawa iterations. This iteration's replacement influence the solution finding process that can be visualized as polynomiographs for the square systems of equations. Polynomiographs presented in the paper, in some sense, are generalization of Kalantari's polynomiography from a single polynomial equation to the square systems of equations. They are coloured based on two colouring methods: basins of attractions with different colours for every real root and colouring dependent on the number of iterations. Possible application of the presented method can be addressed to computer graphics where aesthetic patterns can be used in e.g. texture generation, animations, tapestry design.



### Keywords

Mann iteration, Ishikawa iteration, polynomiography, computer graphics


## 1 INTRODUCTION

Kalantari [Kal05a, Kal08] defined polynomiography as the art and science of visualization in approximation of the zeros of complex polynomials via fractal and non–fractal images created using mathematical convergence properties of iteration functions. As iteration functions the well–known Newton method, methods from Basic Family and Euler–Schröder Family of Iterations can be used. The polynomiograph is a single image that presents visualization process of roots finding for some polynomial. Polynomiographs are two–dimensional images generated in complex plane. Polynomiography, as a method producing nicely looking graphics was patented by Kalantari in the USA in 2005 [Kal05a]. Moreover, it found applications in: creating paintings, carpet design, tapestry design, animations etc. [Kal05b].

In [GKL14, GKL15] the authors presented a survey of some modifications of Kalantari's polynomiography based on the classic Newton's and the higher order Newton-like root finding methods for complex polynomials. Instead of the standard Picard's iteration several different iteration processes were used. By combining different kinds of iterations, different convergence tests, and different colouring they obtained a great variety of polynomiographs [GKL15].



In this paper we discuss a possibility of using the root finding visualization process for square systems of equations with two variables. For such a system the Newton–Raphson method [CK08, SF10] with standard Picard iteration is applicable and works well. In the proposed modification we replaced Picard iteration by Mann and Ishikawa ones. We do not investigate properties of numerical methods after the change of iterations. Mann and Ishikawa iterations are used to perturb the shape of polynomiographs and make them to look more interesting and aesthetically pleasing. So, the main aim of the paper is only to create artistic images.

Usually, for the Newton–Raphson method several iterations are needed to obtain a good accuracy of roots finding approximations. It should be mentioned that the Newton–Raphson method is applicable to more general case – to square systems of equations with any finite number of variables. But visualizations will be presented only in two dimensions, in the plane with two real axes. So, in the sequel, only systems of two equations with two variables are taken into account. This limitation is the first drawback of the method. The second one is the fact that polynomiographs for square systems of equations with two variables can visualize only the real roots of the square systems.

It should be pointed out that the full control of polynomiograph is possible only for the case if the square system has only real roots. Such a situation occurs e.g. for the system:

$$\begin{cases} x(y-1)(x-1)=0, \\ y(y+1)(x+1)=0 \end{cases} \tag{1}$$

having the following five real solutions:

$$\{0,0\},\{0,-1\},\{1,0\},\{1,-1\},\{-1,1\}.$$

The paper is organized as follows. In Sec. 2 Mann and Ishikawa iterations are defined. Sec. 3 presents formulas for Newton–Raphson method and their generalizations obtained using Mann and Ishikawa iterations instead of the standard Picard iteration. In Sec. 4 some examples of polynomiographs are presented. Sec. 5 concludes the paper and shows the future research directions.

## 2 ITERATIONS

Let $w: X \to X$ be a mapping on a metric space $(X,d)$, where $d$ is a metric. Further, let $u_0 \in X$ be a starting point. Following [Ber07] we recall some popular iterative procedures.

- Picard iteration:

$$u_{n+1} = w(u_n), \quad n = 0,1,2,\dots, \tag{2}$$

- Mann iteration:

$$u_{n+1} = \alpha_n w(u_n) + (1-\alpha_n)u_n, \quad n = 0,1,2,\dots, \tag{3}$$

  where $\alpha_n \in (0,1]$.

- Ishikawa iteration:

$$\begin{aligned} u_{n+1} &= \alpha_n w(v_n) + (1-\alpha_n)u_n, \\ v_n &= \beta_n w(u_n) + (1-\beta_n)u_n, \quad n = 0,1,2,\dots, \end{aligned} \tag{4}$$

  where $\alpha_n \in (0,1]$ and $\beta_n \in [0,1]$.

It is easily seen that the Ishikawa iteration with $\beta_n = 0$ for $n = 0,1,2,\dots$ is Mann iteration, and for $\beta_n = 0$, $\alpha_n = 1$ for $n = 0,1,2,\dots$ is Picard iteration. The Mann iteration with $\alpha_n = 1$ for $n = 0,1,2,\dots$ is Picard iteration.

The standard Picard iteration is used in the Banach Fixed Point Theorem [Ber07] to ensure the existence of the fixed point $x^*$ such that $x^* = w(x^*)$ and its approximation under additional assumptions on the space $X$ that should be a Banach one and the mapping $w$ should be contractive. The Mann [Man53] and Ishikawa [Ish74] iterations allow to weak the assumptions on the mapping $w$.

Our further considerations will be conducted in the space $X = \mathbb{R}^2$ that is obviously a Banach one. We take $u_0 = [x_0, y_0]^T \in \mathbb{R}^2$ and $\alpha_n = \alpha$, $\beta_n = \beta$, such that $\alpha \in (0,1]$ and $\beta \in [0,1]$.

## 3 NEWTON–RAPHSON METHOD AND ITS GENERALIZATIONS FOR TWO EQUATIONS WITH TWO UNKNOWNS

By square systems we understand systems with as many equations as variables. Take the following system of non-linear equations:

$$\begin{cases} f(x,y) = 0, \\ g(x,y) = 0, \end{cases} \tag{5}$$

where $f,g: \mathbb{R}^2 \to \mathbb{R}$ and $x$, $y$ are variables.

System (5) can be represented in the form of a single vector equation:

$$\mathbf{F}(x,y) = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0}. \tag{6}$$

We use bold symbols to denote vectors. Assume that $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous function that has continuous first partial derivatives with respect to $x$ and $y$. To solve equation $\mathbf{F}(x,y) = \mathbf{0}$ one can use the Newton–Raphson method [CK08, SF10] starting from an initial point $\mathbf{z}_0 = [x_0, y_0]^T$:

$$\mathbf{z}_{n+1} = \mathbf{z}_n - \mathbf{J}^{-1}(\mathbf{z}_n)\mathbf{F}(\mathbf{z}_n), \quad n = 0,1,2,\dots, \tag{7}$$

where

$$\mathbf{J}(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \\ \frac{\partial g}{\partial x}(x,y) & \frac{\partial g}{\partial y}(x,y) \end{bmatrix} \tag{8}$$

is the $2 \times 2$ Jacobian matrix of $\mathbf{F}$ and $\mathbf{J}^{-1}$ is the inverse matrix to $\mathbf{J}$, that in the case of $2 \times 2$ matrix is given by the following formula:

$$\begin{aligned} \mathbf{J}^{-1}(x,y) = &\frac{1}{\frac{\partial f}{\partial x}(x,y)\frac{\partial g}{\partial y}(x,y) - \frac{\partial f}{\partial y}(x,y)\frac{\partial g}{\partial x}(x,y)} \cdot \\ &\cdot \begin{bmatrix} \frac{\partial g}{\partial y}(x,y) & -\frac{\partial f}{\partial y}(x,y) \\ -\frac{\partial g}{\partial x}(x,y) & \frac{\partial f}{\partial x}(x,y) \end{bmatrix}. \end{aligned} \tag{9}$$

Introducing the operator $\mathbf{N}(\mathbf{z}) = \mathbf{z} - \mathbf{J}^{-1}(\mathbf{z})\mathbf{F}(\mathbf{z})$ we can represent the Newton–Raphson method in the following short form:

$$\mathbf{z}_{n+1} = \mathbf{N}(\mathbf{z}_n), \quad n = 0,1,2,\dots. \tag{10}$$

From this form of Newton–Raphson method we clearly see that the method uses Picard iteration.

Applying the Mann iteration (3) in (10) we obtain the following formula:

$$\mathbf{z}_{n+1} = \alpha \mathbf{N}(\mathbf{z}_n) + (1-\alpha)\mathbf{z}_n, \quad n = 0,1,2,\ldots, \tag{11}$$

where $\alpha \in (0,1]$.

Using the Ishikawa iteration (4) in (10) we get:

$$\begin{aligned} \mathbf{z}_{n+1} &= \alpha \mathbf{N}(\mathbf{v}_n) + (1-\alpha)\mathbf{z}_n, \\ \mathbf{v}_n &= \beta \mathbf{N}(\mathbf{z}_n) + (1-\beta)\mathbf{z}_n, \quad n = 0,1,2,\ldots, \end{aligned} \tag{12}$$

where $\alpha \in (0,1]$ and $\beta \in [0,1]$.

Replacement of the Picard iteration by Mann or Ishikawa iterations leads to the new root finding formulas (11) and (12) that are generalizations of the Newton–Raphson method (7). They produce sequences that if convergent, are convergent to any root of $\mathbf{F}$. This follows from the Hahn–Banach Fixed Point Theorem [Ber07]. Formulas (11) and (12) still produce roots finding sequences but with different character of covergence.

The sequence $\{\mathbf{z}_n\}_{n=0}^{\infty}$ (or orbit of the point $\mathbf{z}_0$) converges or not to a root of $\mathbf{F}$. If the sequence converges to a root $\mathbf{z}^*$ then we say that $\mathbf{z}_0$ is attracted to $\mathbf{z}^*$. A set of all starting points $\mathbf{z}_0$ for which $\{\mathbf{z}_n\}_{n=0}^{\infty}$ converges to $\mathbf{z}^*$ is called the basin of attraction of $\mathbf{z}^*$. Boundaries between basins usually have fractal character due to chaotic behaviour of iteration processes. A good example of such situation can be observed while solving the equation $z^3 - 1 = 0$ in complex plane. Investigations of that case directly led to discovery of Julia and Mandelbrot sets [Man83].

To render polynomiograph for system (5) we can use Algorithm 1. As we noticed earlier the Ishikawa iteration is the most general iteration from the considered set of iterations (Picard, Mann and Ishikawa). So, in the algorithm we use the Ishikawa iteration for the Newton–Raphson method (12), and we denote it by $I_{\alpha,\beta}$. In line 8 of the algorithm we see that we need to determine the colour of the starting point. This could be done in very different ways. In the paper we use two methods: basins of attractions and colouring basing on the iteration (iteration colouring). In the first method to each distinct solution of the system we assign a colour. To determine the colour of the starting point we find the closest solution for the last approximation $\mathbf{z}_{n+1}$ and use its colour. In the second method we have a colourmap (table with colours). Now, to determine the colour of the starting point we take the iteration number for which we have left the while loop and map it to index in the colourmap. To map iterations to indices we used linear interpolation.

**Algorithm 1:** Rendering of polynomiograph for system of equations

**Input**: $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ – left side of system (6), $A \subset \mathbb{R}^2$ – area, $N$ – number of iterations, $\varepsilon$ – accuracy, $\alpha$, $\beta$ – parameters of Ishikawa iteration $I_{\alpha,\beta}$.
**Output**: Polynomiograph for the area $A$.

1 **for** $\mathbf{z}_0 \in A$ **do**
2     $n = 0$
3     **while** $n \leq N$ **do**
4         $\mathbf{z}_{n+1} = I_{\alpha,\beta}(\mathbf{z}_n)$
5         **if** $\|\mathbf{F}(\mathbf{z}_{n+1})\| < \varepsilon$ **then**
6             break
7         $n = n+1$
8     Determine colour for $\mathbf{z}_0$ and print it with the colour

## 4 EXAMPLES

In this section we present some polynomiographs for a system of two equations:

$$\begin{cases} x^3 - y = 0, \\ y^3 - x = 0 \end{cases} \tag{13}$$

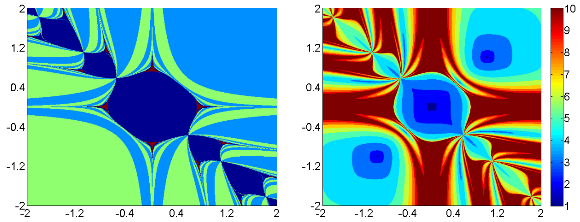
generated using Newton–Raphson method with Picard, Mann and Ishikawa iterations. System (13) has the following four solutions, three of them are real ones and one is complex:

$$\begin{gathered} \{0,0\}, \{1,1\}, \{-1,-1\}, \\ \{0.7071067812 + 0.7071067812i, \\ -0.7071067812 + 0.7071067812i\}. \end{gathered}$$

The common parameters used in the rendering algorithm for all the examples were the following: $A = [-2,2]^2$, $N = 10$, $\varepsilon = 0.001$. The number of points generated in $A$ to obtain the images was set to 600 in each direction.

The examples start with the polynomiographs for the standard Newton–Raphson method, i.e., method with Picard iteration. Fig. 1 presents obtained images. In Fig. 1(a) we see basins of attraction, and in Fig. 1(b) polynomiograph rendered with the iteration colouring (the colourmap is drawn on the right). From the example we see that using the same iteration but different colouring methods we can obtain diverse patterns of polynomiographs.

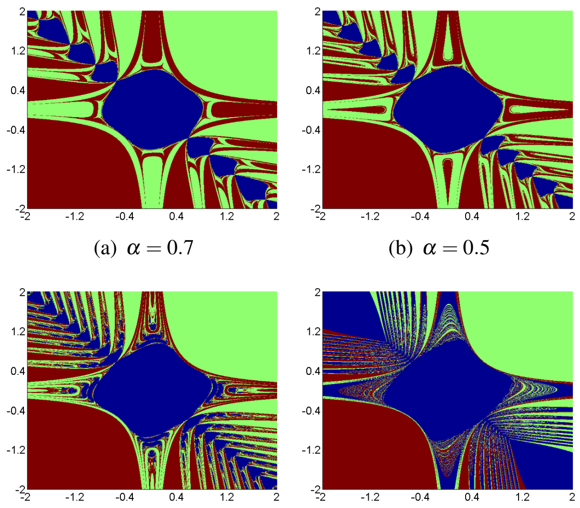
In the second example we use the Mann iteration in the Newton–Raphson method. Fig. 2 presents examples obtained using the basins of attraction colouring method with the following values of $\alpha$ parameter in the Mann iteration: (a) 0.7, (b) 0.5, (c) 0.3, (d) 0.1. Examples showing the use of Mann iteration with iteration colouring are presented in Fig. 3. The values of



(a) basins of attraction (b) iteration colouring

Figure 1: Polynomiographs for system (13) using Picard iteration.

the $\alpha$ parameter were the following: (a) 0.9, (b) 0.8, (c) 0.7, (d) 0.6. In both cases we see that with the change of $\alpha$ the shape of the polynomiograph changes and their shape is different from the polynomiographs obtained with the Picard iteration (Fig. 1). More interesting changes are noticeable in the case of iteration colouring.



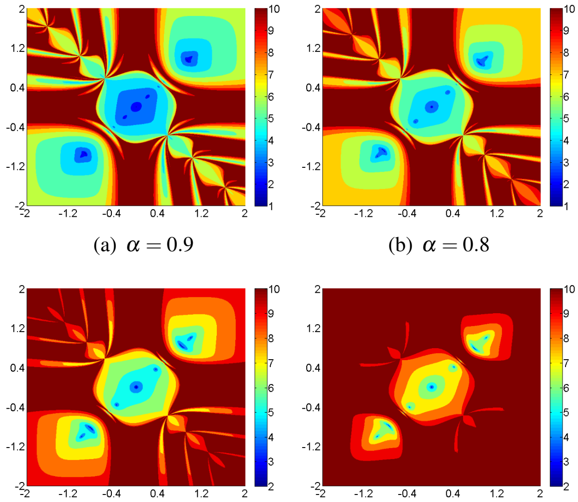
(a) $\alpha = 0.7$ (b) $\alpha = 0.5$

(c) $\alpha = 0.3$ (d) $\alpha = 0.1$

Figure 2: Basins of attraction for system (13) using Mann iteration.

The last example presents the use of Ishikawa iteration in the Newton–Raphson method for system (13). Similar to the case of Mann iteration we generated polynomiographs using two different colouring methods: basins of attraction (Fig. 4) and iteration colouring (Fig. 5). In Fig. 4 we used the following values of the parameters: (a) $\alpha = 0.2$, $\beta = 0.8$, (b) $\alpha = 0.3$, $\beta = 0.7$, (c) $\alpha = 0.7$, $\beta = 0.3$, (d) $\alpha = 0.8$, $\beta = 0.2$, and in Fig. 5 the values were following: (a) $\alpha = 0.6$, $\beta = 0.1$, (b) $\alpha = 0.6$, $\beta = 0.7$, (c) $\alpha = 1.0$, $\beta = 0.7$, (d) $\alpha = 0.7$, $\beta = 0.3$. From the obtained images we clearly see that when we change the parameters values of the Ishikawa iteration we are able to generate a variety of interesting patterns different from those generated with the standard Picard iteration.

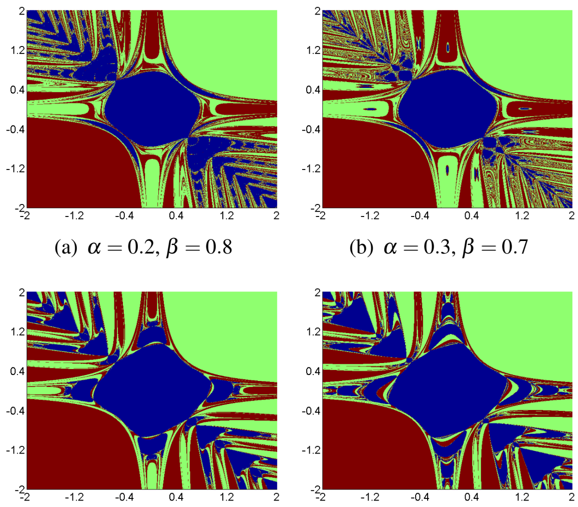
Moreover, looking at Fig. 1(a) and Figs. 2, 4 we can observe that the basins of attraction for each of the three



(a) $\alpha = 0.9$ (b) $\alpha = 0.8$

(c) $\alpha = 0.7$ (d) $\alpha = 0.6$

Figure 3: Polynomiographs for system (13) using Mann iteration and iteration colouring.

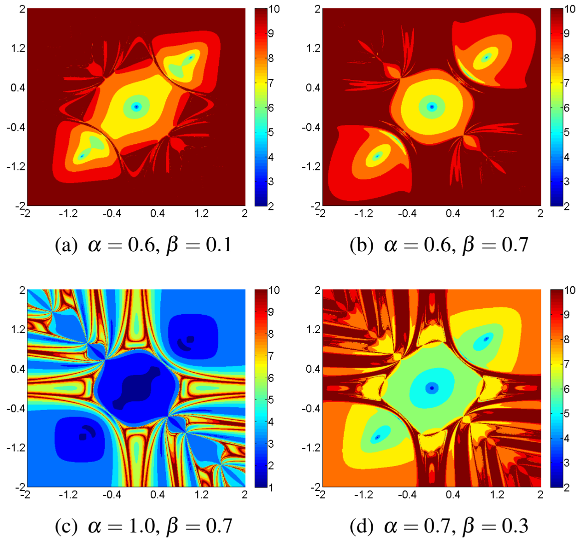


(a) $\alpha = 0.2, \beta = 0.8$ (b) $\alpha = 0.3, \beta = 0.7$

(c) $\alpha = 0.7, \beta = 0.3$ (d) $\alpha = 0.8, \beta = 0.2$

Figure 4: Basins of attraction for system (13) using Ishikawa iteration.

real roots have significantly changed. Some of them have enlarged and other have been divided into many smaller areas, e.g., Fig. 2(d). Thus, using different values of iterations' parameters for some starting points we are able to converge to different roots. Now, looking at Fig. 1(b) and Figs. 3, 5 we can observe how fast the algorithm has found the roots – speed of convergence. The more red colour in the polynomiograph the slower the algorithm. In most of the cases the convergence of the algorithm with the use of Mann and Ishikawa iteration was slower than with the use of Picard iteration. But there are also cases where we see that for the Mann and Ishikawa iteration the red areas in comparison to the Picard iteration have shrunk and the blue areas have become more darker, e.g., Fig. 5(c), so the speed of convergence is faster. Generally, the change of



(a) $\alpha = 0.6, \beta = 0.1$ (b) $\alpha = 0.6, \beta = 0.7$

(c) $\alpha = 1.0, \beta = 0.7$ (d) $\alpha = 0.7, \beta = 0.3$

Figure 5: Polynomiographs for system (13) using Ishikawa iteration and iteration colouring.

speed depends on the iteration used and the value of its parameters.

## 5 CONCLUSIONS AND FUTURE WORK

In the paper we presented some generalizations of the classic Newton–Raphson method using Mann and Ishikawa iterations instead of Picard iteration. These generalizations were then applied to a root finding process for square systems of two equations with two unknowns. Obtained different polynomiographs show a great variety of basins of attractions and images presenting speed of convergence for different iterations.

The results of the paper can be further modified in many directions by the usage of multiparameter iterations, different convergence criteria, different colour maps as e.g. in [GKL14, GKL15]. We can also use other colouring methods or other rendering algorithms of polynomiographs, e.g. algorithms presented in [Gda14]. Moreover, we can try to extend the ideas of the paper related to the use of different iterations, visualization methods of the solution finding process to systems with any number of equations and variables.

We believe that results of the paper can be interesting for those whose work or hobbies are related to automatically creating nicely looking graphics. Also we think that they can be applied to increase functionality of existing polynomiography software.

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