Constrained Shepard Method for Modeling and Visualization of Scattered Data

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ABSTRACT

The problem of modeling and visualization of scattered data where there are inherent constraints on value of the data exists in many scientific and business research areas. For example, the value of mass concentration always has the lower bound of 0 and upper bound of 1. The modeling functions having gradient continuity usually do not guarantee to preserve the bounds of data. In this paper we present the Constrained Shepard method for interpolation of scattered data satisfying the lower and upper bounds specified by the two constraint functions. The constrained interpolant is an extension of the Modified Quadratic Shepard method with comparable efficiency and accuracy. The proposed method is easy to implement and extend to higher dimensionality. The constrained interpolation function is C^1 continuous.

Keywords

Visualization, scattered data interpolation, Shepard's method, bounds, positivity, constraints

1. INTRODUCTION

Data visualization is an important tool used for study of phenomena in scientific and business research. Standard visualization tools require input of data at the specified grids. However, this is not always possible to collect data at the required grids due to various constraints (e.g. economical, physical, temporal, socio-political etc.). To visualize the scattered data, it is required first to construct a faithful model of the reality represented by the data samples. The model is then used to approximate the reality at the required grids. Interpolation and approximation methods are usually used for this purpose. Many methods are available for modeling the reality from the scattered data samples. The surveys [Fra91], [Lod99] and [Ami02] provide good reference to the scattered data modeling methods and their applications. These methods differ in capabilities and characteristics. In general, there is no single best method for all application areas. Requirements of an application determine the suitability of an existing interpolation method. Careful selection from the existing methods suiting the application is often required. An existing modeling method may not have all the characteristics required for the application. We some times need extension in the existing method to incorporate the underlying constraints of the data. A few examples of such constraints are positivity, monotonicity, convexity, gradient and bounds that are commonly encountered in various scientific and business applications. These are usually the known facts about the reality being modeled. The modeling function must not produce results that contradict such known facts about the data. Otherwise the reality discovered using the model may not be trustworthy. Work of many researchers has been reported in literature to preserve the above mentioned constraints of the data. We refer to the work [Sar00] and [But91] for preservation of monotonicity of data on regular grids and [Han97] for the scattered data. We refer to the work of Schmidt [Sch90] for preservation of convexity, monotonicity and positivity of the data on regular grids.

Non-negative, fractional and percentage values are commonly encountered in many areas of science, engineering and business. It does not make sense if the stated values of mass, volume, number of persons and radiation dose are negative. The percentage mass concentration is meaningless if it is below zero or above 100. Many researchers have worked to solve the problem of positivity with various interpolation methods. For related literature and background we refer to the work reported in [Nad92], [Sar00a], [Bro93] and [Mas96].

Shepard's family of interpolation methods introduced by Shepard [She68] is commonly used for metric interpolation of large sets of multidimensional scattered data. The multidimensional datasets are commonly encountered in the fields of business, science and engineering research. A smooth function of the Shepard's interpolation family, known as the Modified Quadratic Shepard (MQS) method, has excellent efficiency and accuracy characteristics. This method can be easily extended to any dimensionality. However, it does not satisfy the constraints imposed by various applications. Work has been reported in [Asi00], [Bro05] and [Asi04] for constrained interpolation of scattered data using the MQS method. These extensions to the MQS method are computationally expensive especially for large and multi-dimensional datasets. The suggested methods also reduce accuracy or continuity of the interpolant. In this paper we present a method and refer it as the Constrained Shepard method that preserves the upper and lower bounds of data specified by the two constraint functions. The constrained interpolation function is C^1 continuous. This method is better than the previous extensions in accuracy, efficiency and extendibility.

The rest of the paper is organized as follows: An overview of the Shepard family of interpolants and need for extension of the work is given in section 2. The Constrained Shepard method is presented in section 3. The advantages and limitations of the method are demonstrated and discussed in section 4. In section 5, we concluded and gave future directions of the research.

2. AN OVERVIEW OF THE MODIFIED SHEPARD METHODS

Let a set of N non-negative data values f_i at the associated scattered sampling locations $X_i = (x_{1i}, x_{2i}, \ldots\ldots, x_{mi})'$, where m is the number of independent variables and $i = 1, 2, \ldots\ldots, N$, are given. The interpolation method due to Shepard [She68] is defined as follows:

$$\overline{F}(X) = \frac{\sum_{i=1}^{N} w_i(X) \quad f_i}{\sum_{i=1}^{N} w_i(X)} \qquad (1)$$

Where, $w_i(X) = \dfrac{1}{d_i^2(X)}$

and $d_i(X) = |X - X_i|_2$ is the radial distance from position X to X_i.

The interpolation function $\overline{F}(X)$ has many interesting properties. The method is easy to implement and extend to higher dimensionality. There is no setup up time for the interpolant. The interpolation function is translation, rotation and scale invariant. This is a global method where each sample value represents. For example it is bounded between the maximum and minimum values in the dataset [Gor78]. Although this interpolant is bounded between maximum and minimum values in the dataset i.e. it satisfies the bounds, however this is sometimes an unnecessary and misleading characteristic for visualization applications. The gradient of the interpolant at each of the data points is zero, as shown in Figure 1, which too is misleading for many visualization applications. As this is a global method so it becomes inefficient for large datasets.

A number of modifications have been suggested to overcome the drawbacks of the Shepard's method. We will focus only on the few modifications of interest for visualization of multidimensional data. The modification by Franke and Neilson [Fra80] improved continuity of the interpolant that replaced constant basis function f_i in Eq. (1) by the quadratic basis function $Q_i(X)$ defined as follows:

$$Q_i(X) = f_i + g_i^T (X - X_i) + \frac{1}{2}(X - X_i)^T A_i (X - X_i) \qquad (2)$$

The matrix A_i in Eq. (2) is the Hessian matrix of the quadratic basis function $Q_i(X)$ and g_i^T is the gradient vector. The quadratic basis function $Q_i(X)$ has following characteristics:

(1). The $Q_i(X_i) = f_i$ i.e. the $Q_i(X)$ interpolates the corresponding data value.

(2). The $Q_i(X)$ is an inverse distance weighted least square approximation to the N_q nearest data points.

The resulting quadratic Shepard interpolation function $F(X)$ is defined as follows:

$$F(X) = \frac{\sum_{i=1}^{N} w_i(X) Q_i(X)}{\sum_{i=1}^{N} w_i(X)} \qquad (3)$$

The modification given above not only improves continuity but also accuracy of the interpolant. Franke and Neilson [Fra80] have proved that the interpolation function $F(X)$ in Eq. (3) is C^1 continuous.

To overcome the inefficiency of the Shepard's method, which is due to its global nature, Franke and Neilson [Fra80] defined the following weight functions:

$$w_i(X) = \left[\frac{R - d_i(X)_+}{R d_i(X)}\right]^2$$

Where $[R - d_i(X)]_+ = \begin{cases} R - d_i(X) & \text{if} \quad R > d_i(X) \\ 0 & \text{otherwise} \end{cases}$

and $R = R_q$ is the radius within which the nodes take part in the construction of $Q_i(X)$ and $R = R_w$ for evaluation of the $F(X)$. Franke and Neilson suggested following formula to evaluate the values of R_q and R_w for a dataset of size N:

$$R_q = \frac{D}{2}\sqrt{\frac{N_q}{N}} \quad \text{, and} \quad R_w = \frac{D}{2}\sqrt{\frac{N_w}{N}}$$

Where $D = \max_{i,j}\left|X_i - X_j\right|_2$ and N_q is the number of data points used for construction of the least square quadratic $Q_i(X)$ and N_w is the number of data points used to evaluate the $F(X)$. The constant D for a dataset is the maximum distance between two points in the dataset. The suggested values for evenly distributed *2D* data are $N_w = 9$ and $N_q = 18$. For sparse data or where datasets are small (i.e. $N<25$) considerable increase in the numbers N_q and N_w is suggested with constant ratio of $N_q/N_w=2$. Renka [Ren88] obtained improvement in accuracy using separate R_{iw} and R_{iq} values for each of the data points and used different criteria for their evaluation. The R_{iq} and R_{iw} in the method suggested by Renka are the smallest radii that enclose the nearest N_q and N_w data points respectively. The suggested values [Ren88] for *2D* data are $N_w=19$ and $N_q=13$.

The MQS interpolation function defined above has excellent efficiency and accuracy characteristics. This is a C^1 continuous function which is easy to implement and extend to higher dimensionality. These characteristics make the method a suitable choice for efficient modeling and visualization of large sets of multidimensional data. However, it is not suitable for applications where there are some inherent constraints on value of the data. Examples of such constraints are non negativity, upper and/or lower bounds and more general constraint defined by a function. Such constraints commonly arise in science and engineering applications. Positivity is a special case of the generalized bounds preserving problem. Examples of the datasets imposing this constraint are mass, volume and density that are always positive. The problem of preserving arbitrary lower bound also exists in science and engineering applications. For example temperature measured on Celsius scale must preserve the lower bound of absolute zero (-273.15 C^o). Gauge pressure should not be less than the negative of atmospheric pressure which is function of the altitude position. This is an example of the lower bound defined by a function. Similarly the problem of preservation of both upper and lower bounds is also common in business, science and engineering. For example: mass and volume concentration must lie between 0 and 1. A value below zero or above 1 is meaningless in such cases. The problem of arbitrary upper and lower bounds preservation are common in business and engineering optimization.

Time (sec)	0	20	40	100	280	300	320
Oxygen (%)	20.8	8.8	4.2	0.5	3.9	6.2	9.6

Table 1. Oxygen levels in flue gases from a boiler.

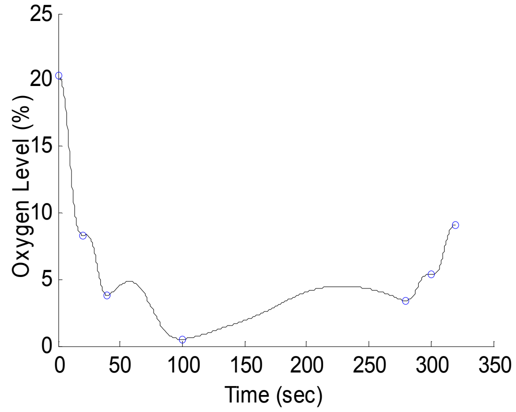

Figure 1. Graph through the data in Table 1 using the Shepard's method

The MQS method does not guarantee to preserve such bounds of data. Samples of the oxygen mass concentration in flue gases from a boiler with respect to time [Asi00] are given in Table 1. A graph has been constructed in Figure 2 through the dataset using the MQS method. The interpolated negative mass concentration values in the graph do not make sense. So, we need an interpolation method that efficiently preserves the above given inherent constraints of the datasets encountered in various application areas.

3. THE CONSTRAINED SHEPARD METHOD

The basis functions of the MQS method, defined by Eq. (2) are inverse distance weighted least square quadratic approximations. It is due to these best fitted basis functions that the MQS method has good data modeling accuracy. The Shepard's interpolant defined in Eq. (1) is bounded between the maximum and minimum values in the dataset [Gor78].

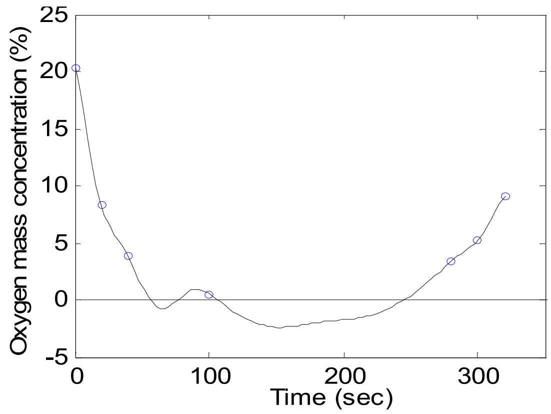

Figure 2. Interpolated negative values of mass concentration using the MQS method.

Similarly the MQS interpolation function is bounded between the maximum and minimum values of the contributing basis functions in their domains of participation. We used these facts to have an accurate constrained interpolation method. The constrained method is based on the maxima and minima principle of the Shepard's family. We used constrained basis functions $\hat{R}_i(X)$ which are piecewise continuous functions approximating the corresponding basis functions of the MQS method and satisfying the upper and lower constraints.

Let $C_U(X)$ and $C_L(X)$ are the functions defining the upper and lower bounds respectively. To construct the constrained basis function, we define difference functions $D_U(X)$ and $D_L(X)$ as the difference between the basis function $Q_i(X)$, defined in (2), and the constraints $C_U(X)$ and $C_L(X)$, respectively i.e.:

$D_{Ui}(X) = [Q_i(X) - C_U(X)]$ and $D_{Li}(X) = [Q_i(X) - C_L(X)]$

Let us rewrite the basis function $Q_i(X)$ as:

$Q_i(X) = f_i + Q_{0i}(X)$

The scaled difference functions, $\hat{D}_{Ui}(X)$ and $\hat{D}_{Li}(X)$, are constructed using the fixed point scaling that maintains values of the difference functions at the data point X_i. The scaled difference functions are defined as follows:

$$\hat{D}_{Ui}(X) = d_{Ui} + K\,[Q_{0i}(X)] \tag{4}$$

and

$$\hat{D}_{Li}(X) = d_{Li} + K\,[Q_{0i}(X)] \tag{5}$$

where $d_{Ui}=f_i-C_U(X_{mU})$ and $d_{Li}=f_i-C_L(X_{mL})$. The X_{mU} represent the point of maximum of the $D_U(X)$ and X_{mL} is the point of minimum of the $D_U(X)$ in the domain of participation of the $Q_i(X)$. For simplicity we will use X_m to represent both X_{mU} and X_{mL}. The coefficient K in (4) and (5) is a positive constant which has value between zero and 1. The constrained basis function $\hat{R}_i(X)$ is defined as follows:

$$\hat{R}_i(X) = \begin{cases} C_U(X_m) + \mu_{Ui}(X)\left[\hat{D}_{Ui}(X)\right] & \text{if} \quad \hat{Q}(X) \ge f_i \\ C_L(X_m) + \mu_{Li}(X)\left[\hat{D}_{Li}(X)\right] & \text{otherwise} \end{cases} \tag{6}$$

The functions $\mu_{Li}(X)$ *and* $\mu_{Ui}(X)$ in Eq. (6) are defined as follows:

$$\mu_{Li}(X) = \begin{cases} \left(\dfrac{\hat{D}_{Li}(X)}{d_{Li}}\right)^n & \text{if} \quad C_L \le \hat{Q}_i(X) \le f_i \\ 0 & \text{Otherwise} \end{cases} \tag{7}$$

and

$$\mu_{Ui}(X) = \begin{cases} \left(\dfrac{\hat{D}_{Ui}(X)}{d_{Ui}}\right)^n & \text{if} \quad f_i \le \hat{Q}_i(X) \le C_U \\ 0 & \text{Otherwise} \end{cases}$$

where $\hat{Q}_i(X) = f_i + K.Q_{0i}(X)$ (8)

and $$n = \frac{1}{K} - 1 \tag{9}$$

The Constrained Shepard interpolation function is defined as follows:

$$\hat{F}(X) = \frac{\sum_{i=1}^{N} w_i(X)\hat{R}_i(X)}{\sum_{i=1}^{N} w_i(X)} \tag{10}$$

The weight functions $w_i(X)$ of the Constrained Shepard method in Eq. (10) are same as defined by Renka [Ren88]. The maximum value of K for which the $\hat{Q}_i(X)$ in Eq. (8) is constrained between the bounds is the best value of K factor for the basis function. However in this research we use a constant value of K for all the basis functions for efficiency reasons. The valid range for K value is $0 < K < 1$. We use K as an input parameter which gives us flexibility to use a value that is suitable for the application. We propose a value of $K = 1/3$ that suits many applications. We used the same value of K for construction of all the examples and comparisons in this research. Similarly we suggest the use of approximate values for $C_U(X_m)$ and $C_L(X_m)$ to reduce computational cost of searching minimum/maximum of each of the difference functions. The minimum of upper constraint and maximum of the lower constraint functions in the whole domain of interest may be used for an efficient solution.

A combination of the values of K and the constraint functions $C_L(X)$ and $C_U(X)$ defines the characteristics of the Constrained Shepard method. Following are a few special cases of the value constrained problems commonly encountered in science and engineering application areas. The corresponding combination of the input values of K, $C_L(X)$ and $C_U(X)$, to handle the cases, using the Constrained Shepard method, are also given.

Case 1: For preserving lower bound 0 and upper bound 1, the $C_U(X_m)=1$ and $C_L(X_m) = 0$. The $\hat{R}_i(X)$ in this case reduces to:

$$\hat{R}_i(X) = \begin{cases} 1 + \mu_{Ui}(X)\left[\hat{Q}_i(X) - 1\right] & \text{if} \quad f_i \le \hat{Q}_i(X) \\ \mu_{Li}(X)\hat{Q}_i(X) & \text{Otherwise} \end{cases}$$

Case 2: Where $\hat{R}_i(X)$ is required to preserve lower bound only, the input value of K is selected between zero and 1. The upper bound function $C_U(X_m)$ is defined by a large constant value. This large value can be estimated by multiplying the maximum value in the data set by a large positive number.

Case 3: Where the basis function $\hat{R}_i(X)$ is required to preserve upper bound only, the input value of K is selected between zero and 1. The lower bound function is defined by a negative constant of large magnitude.

The interpolation function $\hat{F}(X)$ defined in Eq. (10) has the following properties:

Theorem 3.1. For all sample positions i, the interpolant $\hat{F}(X)$ in Eq. (10) satisfies the following for all independent variables x_d where d = 1, 2, ..., m:

$$\left.\frac{\partial \hat{F}(X)}{\partial(x_d)}\right|_{X=X_i} = \left.\frac{\partial \hat{R}_i(X)}{\partial(x_d)}\right|_{X=X_i}$$

Proof: We refer to [Ren88] for proof of the theorem.

Theorem 3.2. If the input value of K=1, the basis function of the Constrained Shepard method degenerates to $\hat{R}_i(X) = Q_i(X)$ between the bounds. The basis function beyond the bounds are equal to the minimum/maximum of the corresponding constraint i.e. $C_L(X_m)$ and $C_U(X_m)$.

Theorem 3.3. At all the positions X, for which $\hat{Q}_i(X) = f_i$, the basis function in Eq. (6) satisfies:

$$\left.\hat{R}_i(X)\right|_{\hat{Q}_i(X)=f_i} = f_i$$

Proof: Let $\hat{R}_{Ui}(X)$ and $\hat{R}_{Li}(X)$ are the lower and upper part of $\hat{R}_i(X)$ defined by the Eq. (6) i.e.:

$$\hat{R}_{Ui}(X) = C_U(X_m) + \mu_{Ui}(X)\hat{D}_{Ui}(X) \tag{11}$$

and

$$\hat{R}_{Li}(X) = C_L(X_m) + \mu_{Li}(X)\hat{D}_{Li}(X)$$

Combining the equations (7) and (11) we get:

$$\hat{R}_{Ui}(X) = C_U(X_m) + \hat{D}_{Ui}^{n+1}(X)/d_{Ui}^n \tag{12}$$

Fixing $\hat{Q}_i(X) = f_i$ in Eq. (12) we get

$$\left.\hat{R}_{Ui}(X)\right|_{\hat{Q}_i(X)=f_i} = f_i$$

We can prove similarly for lower part of $\hat{R}_i(X)$ that

$$\left.\hat{R}_{Li}(X)\right|_{\hat{Q}_i(X)=f_i} = f_i$$

Theorem 3.4. If 0<K<1, the first partial derivatives of $\hat{R}_i(X)$ exist and continue at X_i. Moreover for all independent variables x_d, where d=1, 2, , m:

$$\left.\frac{\partial \hat{R}_i(X)}{\partial x_d}\right|_{X=X_i} = \left.\frac{\partial Q_i(X)}{\partial x_d}\right|_{X=X_i}$$

Proof: Let $\hat{R}_{Ui}(X)$ be the upper part of basis function $\hat{R}_i(X)$ defined in Eq. (12) i.e.:

$$\hat{R}_{Ui}(X) = C_U(X_m) + \hat{D}_{Ui}^{n+1}(X)/d_{Ui}^n$$

The first partial derivative of the above given equation with respect to x_d results in:

$$\frac{\partial \hat{R}_{Ui}}{\partial x_d} = \frac{\partial C_U(X_m)}{\partial x_d} + \frac{(n+1)\hat{D}_{Ui}^n(X)}{d_{Ui}^n}\left[\frac{\partial \hat{Q}_i(X)}{\partial x_d}\right] \tag{13}$$

where as $\frac{\partial \hat{Q}_i(X)}{\partial x_d} = K\frac{\partial Q_i(X)}{\partial x_d}$ (14)

From Eq. (9) $K = \frac{1}{n+1}$

As the $Q_{0i}(X_i)=0$, using this value in Eq. (4) we get:

$$\left.\frac{\hat{D}_{Ui}^n(X)}{d_{Ui}^n}\right|_{X=X_i} = 1 \tag{15}$$

Combining the equations from (13) to (15) we get:

$$\left.\frac{\partial \hat{R}_{Ui}(X)}{\partial x_d}\right|_{X=X_i} = \left.\frac{\partial Q_i(X)}{\partial x_d}\right|_{X=X_i} \tag{16}$$

We can prove similarly for lower part of $\hat{R}_i(X)$ that

$$\left.\frac{\partial \hat{R}_{Li}(X)}{\partial x_d}\right|_{X=X_i} = \left.\frac{\partial Q_i(X)}{\partial x_d}\right|_{X=X_i}$$

Theorem 3.5. If 0<K<1, the first partial derivatives of $\hat{R}_i(X)$ where the $\hat{R}_i(X)$ approaches $C_U(X_m)$ or *the $\hat{R}_i(X)$ approaches $C_L(X_m)$ exist and continues and for all variables x_d where d=1, 2, , m, i.e.:*

$$\left.\frac{\partial \hat{R}_i(X)}{\partial x_d}\right|_{\hat{R}_i(X)=C_U(X_m)} = 0$$
$$\left.\frac{\partial \hat{R}_i(X)}{\partial x_d}\right|_{\hat{R}_i(X)=C_L(X_m)} = 0 \tag{17}$$

Proof: From Eq. (6) the basis function $\hat{R}_i(X)$ approaches $C_U(X_m)$ where the difference function $\hat{D}_{Ui}(X)$ approaches 0. Using the Eq. (13) we can prove that where the $\hat{D}_{Ui}(X)$ approaches 0:

$$\left.\frac{\partial \hat{R}_i(X)}{\partial x_d}\right|_{\hat{D}_{Ui}(X)=0} = \frac{\partial C_U(X_m)}{\partial x_d} = 0$$

Similarly that where $\hat{D}_{Ui}(X)$ approaches 0:

$$\left.\frac{\partial \hat{R}_i(X)}{\partial x_d}\right|_{\hat{D}_{Li}(X)=0} = \frac{\partial C_L(X_m)}{\partial x_d} = 0$$

Theorem 3.6. The interpolation function defined by the Eq. (10) is C^1 continuous.

Proof: To prove that the interpolant is C^1 continuous it is sufficient to prove that the first partial derivatives of the basis function $\hat{R}_i(X)$ in Eq. (10) exists and continuous in the domain of its participation [Ren88]. The theorems 3.1 to 3.5 prove the theorem 3.6.

4. RESULTS AND DISCUSSION

Implementation of the Constrained Shepard method and its extension to higher dimensionality is as simple as that of the MQS method where constant bounds are involved. Only a few changes in the main module of the existing implementation of the MQS method are required. The additional user inputs required in this method are the value of *K* and the two arrays holding the coefficients of the constraint

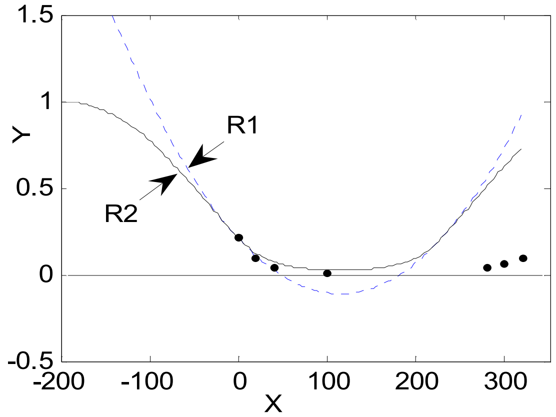

Figure 3. The quadratic basis function (R1) has values greater than 1 and less than 0 while the constrained basis function (R2) remains between 0 and 1.

functions $C_U(X)$ and $C_L(X)$ defining the upper and lower bounds respectively.

In Figures 3 and 4 we have demonstrated that how the convex and concave basis functions of the MQS interpolant are modified to preserve the lower bound of 0 and upper bound of 1. We can observe from the graphs that the constrained basis functions are smooth in the whole domain joining smoothly to the lower & upper bounds. Slope of the constrained basis functions become zero where their value approach zero (at the lower bound that is a constant) or 1 (the upper bound that too is a constant). The constrained basis functions do not depart much from the basis functions of the MQS method especially in the vicinity of their own data points. This characteristic minimizes the negative effect, which may occur due to the departure from the least square fitted basis function, on accuracy of the interpolant. Graphs through the dataset in Table 1 using the MQS and the constrained interpolants are shown in Figure 5. We can observe from the graphs that the constrained interpolation function preserves lower and upper bounds (i.e. 0 and 100 respectively) inherent to the given data. The graph is smooth and a close approximation of the graph due to the MQS method.

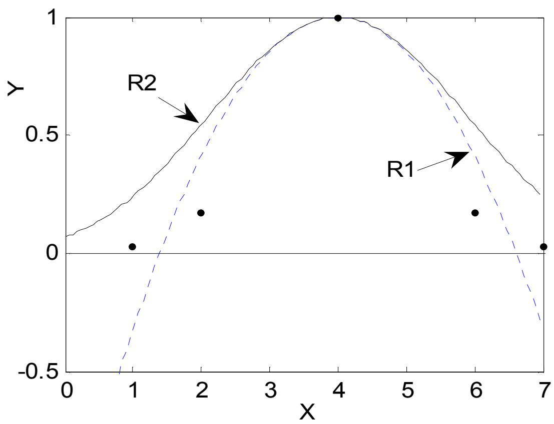

Figure 4. The quadratic basis function (R1) has values less than 0 while the constrained basis function (R2) remains above 0.

To analyze the accuracy and efficiency of the Constrained Shepard method, the *2D* datasets generated using the following test functions were used:

$$F(x,y)=\begin{cases}1 & \text{if} \quad (y-x)\rangle \frac{1}{2} \\ 2(y-x) & \text{if} \quad 0\le (y-x)\le \frac{1}{2} \\ \frac{1}{2}\cos(4\pi P^{\frac{1}{2}})+\frac{1}{2} & \text{if} \quad P\le \frac{1}{16} \\ 0 & \text{Otherwise}\end{cases} \qquad (18)$$

$$\text{where } P=(x-\tfrac{3}{2})^2+(y-\tfrac{1}{2})^2$$

$$F(x,y) = exp(x)\,sin^2(y) \qquad (19)$$

$$F(x,y) = sin^2(x)\,sin^2(y) \qquad (20)$$

The above given functions represent a few natural phenomena i.e. linear, exponential, constant and harmonic etc. The test functions (18) and (20) are bounded between 0 and 1. Data generated at 30 random positions using the test functions has been used for visual comparison and estimation of the deviations and execution time.

The MQS method is known to have excellent efficiency, accuracy and smoothness characteristics and it is easy to implement and extend to higher dimensionality. We will use these characteristics to assess the capabilities of the Constrained Shepard method.

- The MQS interpolation function is C^1 continuous. The C^1 continuity is required for most of the visualization and other applications for reasons like visually pleasing, visual perception and continuity of the phenomenon that the dataset is representing. The constrained interpolation function, we proved in the previous section, is C^1 continuous. Gradients of the MQS and Constrained Shepard interpolants are equal at all the data points which lie within the bounds.

The basis functions of the MQS method are inverse

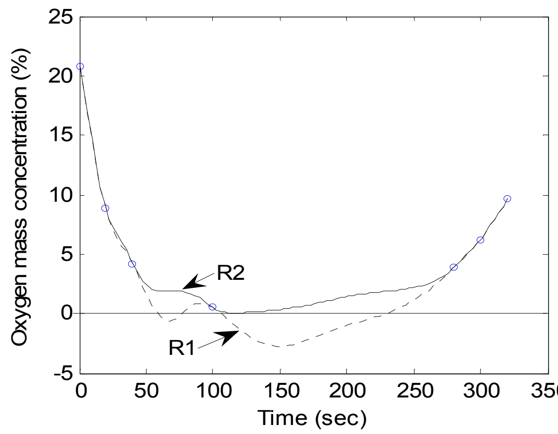

Figure 5. One-dimensional data (Table 1) using the MQS (R1) and the Constrained Shepard (R2) method.

Test functions of the data sets	Performance parameters / measures	Deviations from the test functions		Jackknifing Errors	
		MQS method	Constrained Shepard method	MQS method	Constrained Shepard method
Eq. (18)	RMS	0.1482	0.0995	0.1612	0.1646
	Absolute Maximum	1.287195	0.5059	0.7207	0.7822
Eq. (19)	RMS	0.2452	0.2460	0.0129	0.0101
	Absolute Maximum	0.8856	0.9046	0.0499	0.0336
Eq. (20)	RMS	0.0137	0.0153	0.0182	0.0148
	Absolute Maximum	0.0801	0.0923	0.0479	0.0497
	Execution time (seconds)	0.0175	0.01797		

Table 2. Efficiency and accuracy measures using the MQS and Constrained Shepard methods. Data used is generated at 30 random locations using the test functions. The time is for 25x25 grids execution.

distance weighted least square quadratic functions. Any deviation of the quadratics from the least square fit may results in the increase of deviations of the graph from original. Use of the piecewise continuous constrained basis functions minimizes its deviation from the least square fitted quadratic basis functions of the MQS method. So its accuracy measures are very close to the MQS method as depicted from the measurements of jackknifing errors and the deviations from the test functions. The Root Mean Square (RMS) and Absolute Maximum (AM) deviations of three randomly generated datasets, using the test functions given by Eq. (18) to (20) are listed in the Table 2. The RMS & Absolute Maximum (AM) jackknifing error estimates for the same datasets are also given. The MQS and Constrained Shepard methods have similar accuracy measures as depicted from the Table 2. A dataset generated using the test function (18) is plotted on 25x25 grids, using the MQS method, in the Figure 6. The graph does not preserve the lower and upper bounds i.e. 0 and 1. A graph through the same dataset using the Constrained Shepard method is shown in the Figure 7. The graph is constrained between the bounds i.e. 0 and 1. This graph seems to be a closer approximation of the graph of the test function that is plotted in Figure 8.

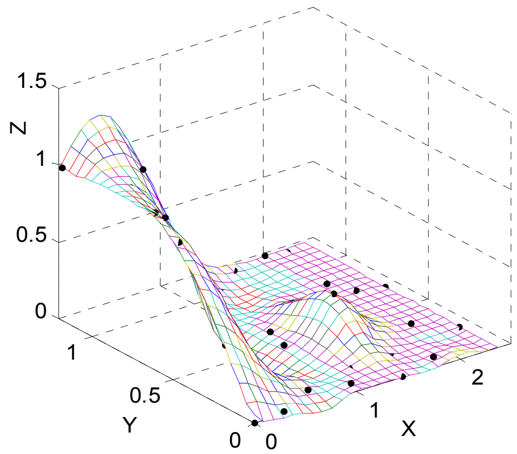

Figure 6. A randomly generated dataset (N=30) using the test function (18) is plotted using the MQS method. The graph does not preserve [0, 1] bounds of data.

The Constrained Shepard method is slightly expensive computationally than the MQS method. The computational time for generation of 25x25 grids using the MQS and the Constrained Shepard method are given in Table 2. Machine used is PC, P-IV, 2.4 GHz; 496 MB RAM with windows XP operating system. Larger the sample size: less will be the relative computational cost of the Constrained Shepard and MQS methods for constant grids. For a

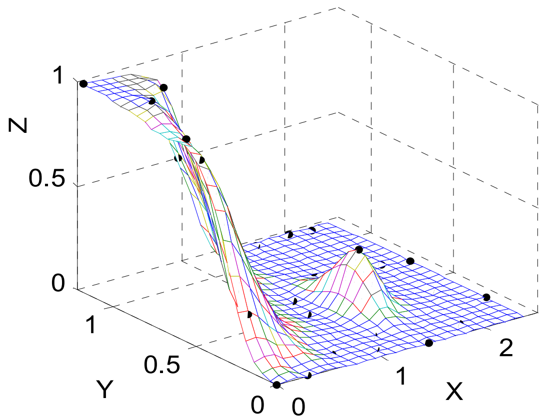

Figure 7. The graph due to the Constrained Shepard method is bounded between 0 and 1.

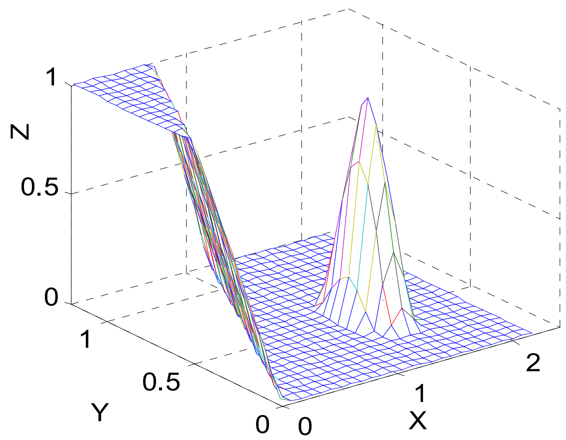

Figure 8. Test function (18) plotted at 25x25 grids.

constant size dataset, the relative cost will increase with increase of the number of grids. The application areas of the interpolant involve very large datasets. The Constrained Shepard method imposes very small efficiency penalty making it a suitable choice for constrained modeling of very large datasets.

The formulation of the Constrained Shepard method, given in this research, is without reference to the number of dimensions of the data. It is easy to implement the Constrained Shepard method for higher dimensional data.

5 CONCLUSIONS & FUTURE WORK

We have presented an efficient method for modeling scattered data where there are inherent constraints on value of the data samples. The method handles the upper and lower constraints while maintaining efficiency, C^1 continuity, accuracy and extendibility of the MQS method. Hopefully this will be a valuable method for constrained modeling and visualization of very large sets of multidimensional scattered data.

Typical application areas of this research are visualization of environmental data, locating mobile target using wireless sensors networks and multidimensional optimization problems in business and engineering i.e. optimization of cellular communication networks.

We are working to implement the method for higher dimensional data for applications in engineering optimization.

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