

Generalization of $(2n + 4)$ -point approximating subdivision scheme

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ABSTRACT

In this paper, we present explicitly a general formula for the mask of $(2n + 4)$ -point approximating subdivision scheme with two parameters which reproduces all polynomials of degree $\leq 2n + 1$. The proposed scheme generalizes several subdivision schemes such as the Chaikin's algorithm, the 4-point approximating scheme and the $(2n + 2)$ -point approximating schemes.

Keywords

Approximating subdivision scheme, mask, convergence and smoothness.

1. INTRODUCTION

Subdivision schemes define a smooth curve or surface as the limit of a sequence of successive refinements. By these methods at each refinement step, new inserted points on a finer grid are computed by affine combination of the already existing points. In the limit of the recursive process, data are defined on a dense set of points.

Each subdivision scheme is associated with a mask $a = \{a_i\}_{i \in \mathbb{Z}}$, where we use \mathbb{Z} to denote the set of all integers. Throughout the work, we consider schemes with a mask of finite support. The binary subdivision scheme is a process which recursively defines a sequence of control points $f^k = \{f_i^k\}_{i \in \mathbb{Z}}$ by a rule of the form with a mask $a = \{a_i\}_{i \in \mathbb{Z}}$

$$f_i^{k+1} = \sum_{j \in \mathbb{Z}} a_{i-2j} f_j^k, \quad k \in \{0, 1, 2, \dots\},$$

which is denoted formally by $f^{k+1} = S f^k = S^k f^0$.

A subdivision scheme is said to be *uniformly convergent* if for every initial data $f^0 = \{f_i\}_{i \in \mathbb{Z}}$, there is a continuous function $f \in C(\mathbb{R})$ such that for any interval $[a, b]$

$$\lim_{k \rightarrow \infty} \sup_{i \in \mathbb{Z} \cap 2^k [a, b]} |f_i^k - f(2^{-k} i)| = 0,$$

and such that $f \neq 0$ for some initial data. We denote the function f by $S^\infty f^0$ and call it a limit function of subdivision scheme S .

There are two ways to obtain a desired mask. One is to enlarge the size of support so that we can allow free parameters in the mask, the other is to fix the size of support and find specific values of the mask satisfying a certain property. There is no unique method of obtaining a mask. Deslauries and Dubuc [2,5] obtained the mask of a $2n$ -interpolatory subdivision scheme reproducing all polynomials of degree $\leq 2n - 1$. Ko et al. [6] rebuilt the masks of interpolatory symmetric subdivision schemes such as 4-point and 6-point interpolatory subdivision schemes, ternary 4-point interpolatory scheme, butterfly scheme and modified butterfly scheme by using only two conditions of symmetry and a necessary condition for smoothness. Ko et al. [7] obtained a general rule for the construction of the mask of $(2n + 4)$ -point symmetric subdivision schemes with two parameters, that reproduces all polynomials of degree $\leq 2n + 1$.

For the further study of Ko et al. [7], in much the same way, we present masks of $(2n + 4)$ -point stationary approximating subdivision scheme with two parameters which reproduces all polynomials of degree $\leq 2n + 1$ and the proposed scheme provides good smoothness. The proposed scheme generalized the masks of well-known approximating subdivision scheme such as Chaikin's algorithm, Dyn et al. [4] 4-point approximating scheme, Choi et al. [1]

proposed scheme and B-spline of degree 6. Choi et al. [1] presented a new class of subdivision schemes. They proved the convergence, smoothness and approximation order of the proposed schemes. But they did not get the explicit formulation for the mask. The proposed subdivision schemes in this work have two parameters. For the advantages of using two parameters, we can get more flexible design of scheme by setting the two parameters to various values. And this new formula of the masks will be useful to analyze the $(2n+4)$ -point approximating subdivision schemes for further study.

2. CONSTRUCTION OF SCHEME

We denote by P_{2n+1} the space of all polynomials of degree $\leq 2n+1$ for a nonnegative integer n . We define the Lagrange fundamental polynomials $\{L_k(x)\}_{k=-n}^{n+1}$ corresponding to the nodes $\{k\}_{k=-n}^{n+1}$ by

$$L_k(x) = \prod_{j \neq k, j=-n}^{n+1} \frac{x-j}{k-j}, \quad k = -n, \dots, n+1, \quad (1)$$

for which

$$L_k(j) = \delta_{k,j}, \quad k, j = -n, \dots, n+1, \quad (2)$$

and

$$\sum_{k=-n}^{n+1} p(k)L_k(x) = p(x), \quad p \in P_{2n+1}. \quad (3)$$

Then it is easy to see that for each $j = -n-1, \dots, n$,

$$L_{-j}\left(\frac{1}{2}\right) = (-1)^j \frac{(n+1)}{2^{4n+1}} \binom{2n+1}{2j+1} \binom{2n+1}{n+j+1}, \quad (4)$$

$$L_{-j}(n+2) = (-1)^{j+n+1} \frac{(2n+2)!}{(n-j)!(n+j+1)!(n+j+2)}, \quad (5)$$

$$L_{-j}(-n-1) = (-1)^{j+n} \frac{(2n+2)!}{(n-j)!(n+j+1)!(n-j+1)}, \quad (6)$$

$$L_{-j}(n+2) + L_{-j}(-n-1) = (-1)^{j+n} \binom{2n+1}{n+j+1} \frac{(2n+2)(2j+1)}{(n+j+2)(n-j+1)}, \quad (7)$$

and

$$L_{-j}\left(\frac{1}{4}\right) = \frac{\prod_{j=-n-1}^n (4j+1)}{(-1)^{n+1-j} 4^{2n+1} (4j+1)(n-j)!(n+j+1)!}, \quad (8)$$

$$L_{-j}\left(\frac{3}{4}\right) = \frac{\prod_{j=-n-1}^n (4j+3)}{(-1)^{n+1-j} 4^{2n+1} (4j+3)(n-j)!(n+j+1)!}. \quad (9)$$

These quantities are crucial to find the explicit form of masks considered in the following process.

We consider the problem of finding masks $a = \{a_j\}_{j=-2n-4}^{2n+3}$ reproducing polynomials of degree $\leq 2n+1$, that is

$$\sum_k a_{j-2k} p(k) = p\left(\frac{2j+1}{4}\right), \quad j \in \mathbb{Z}, p \in P_{2n+1}. \quad (10)$$

Throughout this section, we let $v = a_{2n+2} = a_{-2n-3}$ and $\omega = a_{2n+3} = a_{-2n-4}$, for convenience's sake. Setting $j=0$ in (10) and using (2) and (3), the equation (10) implies

$$\sum_{k=-n-1}^{n+2} a_{-2k} L_{-j}(k) = L_{-j}\left(\frac{1}{4}\right), \quad j = -n-1, \dots, n. \quad (11)$$

We split the summation on the left-hand side of the equation (11) as

$$\begin{aligned} \sum_{k=-n-1}^{n+2} a_{-2k} L_{-j}(k) &= \sum_{k=-n}^{n+1} a_{-2k} L_{-j}(k) + a_{2n+2} L_{-j}(-n-1) + a_{-2n-4} L_{-j}(n+2) \\ &= a_{2j} + a_{2n+2} L_{-j}(-n-1) + a_{-2n-4} L_{-j}(n+2). \end{aligned}$$

Thus we get the explicit form of a_{2j} for $j = -n-1, \dots, n$,

$$a_{2j} = L_{-j}\left(\frac{1}{4}\right) - v L_{-j}(-n-1) - \omega L_{-j}(n+2). \quad (12)$$

Also setting $j=1$ in (10), we get

$$\sum_{k=-n-1}^{n+2} a_{1-2k} L_{-j}(k) = L_{-j}\left(\frac{3}{4}\right), \quad j = -n-1, \dots, n. \quad (13)$$

By splitting the summation on the left-hand side of the equation (13) and applying the relation (2), we get

$$\begin{aligned} \sum_{k=-n-1}^{n+2} a_{1-2k} L_{-j}(k) &= \sum_{k=-n}^{n+1} a_{1-2k} L_{-j}(k) + a_{-2n-3} L_{-j}(n+2) + a_{2n+3} L_{-j}(-n-1) \\ &= a_{4+2j} + a_{-2n-3} L_{-j}(n+2) + a_{2n+3} L_{-j}(-n-1). \end{aligned}$$

Hence we have the explicit form for a_{2j+1}

$$a_{2j+1} = L_{-j}\left(\frac{3}{4}\right) - v L_{-j}(n+2) - \omega L_{-j}(-n-1), \quad (14)$$

for $j = -n-1, \dots, n$.

We can see that the proposed scheme with mask a_{2j} as given in (12) and a_{2j+1} as given in (14) satisfies the polynomial reproducing property up to degree $2n+1$, because this property is the starting point of the construction of the mask as formulated (10).

EXAMPLE 1 For $n=0$, we have the mask

$$\left[\omega, \nu, \frac{1}{4} + \nu - 2\omega, \frac{3}{4} - 2\nu + \omega, \frac{3}{4} - 2\nu + \omega, \frac{1}{4} + \nu - 2\omega, \nu, \omega \right].$$

In case when $\nu = \omega = 0$, it becomes the Chaikin's scheme:

$$\left[\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4} \right].$$

If we set $\omega = -\frac{5}{128}, \nu = -\frac{7}{128}$, we have the mask of Dyn et al. [4] 4-point approximating scheme:

$$\left[\frac{-5}{128}, \frac{-7}{128}, \frac{35}{128}, \frac{105}{128}, \frac{105}{128}, \frac{35}{128}, \frac{-7}{128}, \frac{-5}{128} \right].$$

When we set $\nu = -\frac{3}{32} - \omega$, we get the same mask as Choi et al. [1] proposed for $L = 3$ case:

$$\left[\omega, \frac{-3}{32} - \omega, \frac{5}{32} - 3\omega, \frac{15}{16} + 3\omega, \frac{15}{16} + 3\omega, \frac{5}{32} - 3\omega, \frac{-3}{32} - \omega, \omega \right].$$

Also, when we set $\nu = \frac{7}{64}, \omega = \frac{1}{64}$, this subdivision scheme becomes the B-spline of degree 6 subdivision scheme:

$$\left[\frac{1}{64}, \frac{7}{64}, \frac{21}{64}, \frac{35}{64}, \frac{35}{64}, \frac{21}{64}, \frac{7}{64}, \frac{1}{64} \right].$$

3. ANALYSIS OF SCHEME

For the convergent subdivision scheme S , the corresponding mask $\{a_i\}_{i \in \mathbb{Z}}$ necessarily satisfies

$$\sum_{i \in \mathbb{Z}} a_{2i} = \sum_{i \in \mathbb{Z}} a_{2i+1} = 1. \quad (15)$$

We introduce a symbol called the Laurent polynomial

$$a(z) := \sum_{i \in \mathbb{Z}} a_i z^i$$

of a mask $\{a_i\}_{i \in \mathbb{Z}}$ with finite support. The corresponding symbols play a efficient role to analyze the convergence and smoothness of subdivision scheme.

With the symbol, Dyn [3] provided a sufficient and necessary condition for a uniformly convergent subdivision scheme: For a subdivision scheme S , S is uniformly convergent if and only if there is an integer $L \geq 1$, such that

$$\left\| \left(\frac{1}{2} S_1 \right)^L \right\|_{\infty} < 1,$$

where S_1 is the subdivision scheme associated with the mask q , where $a(z) = \left(\frac{1+z}{2} \right) q(z)$ and satisfying

$$df^k = S_1 df^{k-1}, \quad k = 1, 2, \dots,$$

for the control points $f^k = S^k f^0$ and $df^k = \left\{ (df^k)_i = 2^k (f_{i+1}^k - f_i^k) \right\}_{i \in \mathbb{Z}}$, and the norm $\|S\|_{\infty}$ of a subdivision scheme S with a mask $\{a_i\}_{i \in \mathbb{Z}}$ is defined by

$$\|S\|_{\infty} = \max \left\{ \sum_{i \in \mathbb{Z}} |a_{2i}|, \sum_{i \in \mathbb{Z}} |a_{2i+1}| \right\}.$$

From the given mask

$$a = \left[\omega, \nu, \frac{1}{4} + \nu - 2\omega, \frac{3}{4} - 2\nu + \omega, \frac{3}{4} - 2\nu + \omega, \frac{1}{4} + \nu - 2\omega, \nu, \omega \right],$$

we have

$$a_1 = 2 \left[\omega, \nu - \omega, \frac{1}{4} - \omega, \frac{1}{2} - 2\nu + 2\omega, \frac{1}{4} - \omega, \nu - \omega, \omega \right],$$

where $a_1(z) = \frac{2z}{1+z} a(z)$. It is easy to verify that $a(z)$ and $a_1(z)$ satisfy the necessary condition (15) for the convergence of S and S_1 . If

$$\left\| \frac{1}{2} S_1 \right\|_{\infty} = \max \left\{ 2|\omega| + 2 \left| \frac{1}{4} - \omega \right|, \left| \frac{1}{2} - 2\nu + 2\omega \right| + 2|\nu - \omega| \right\} < 1,$$

then this scheme converges to continuous limit function. We have the mask of S_2 using equation

$$a_2(z) = \frac{2z}{1+z} a_1(z);$$

$$a_2 = 4 \left[\omega, \nu - 2\omega, \frac{1}{4} - \nu + \omega, \frac{1}{4} - \nu + \omega, \nu - 2\omega, \omega \right].$$

If

$$\left\| \frac{1}{2} S_2 \right\|_{\infty} = \max \left\{ 2|\omega| + 2 \left| \frac{1}{4} - \nu + \omega \right| + 2|\nu - 2\omega| \right\} < 1,$$

then this scheme is $C^1(R)$.

For C^2 continuity, $a_2(z)$ should satisfy (15), which is true. From the relation $a_3(z) = \frac{2z}{1+z} a_2(z)$, we have the mask of S_3

$$a_3 = 8 \left[\omega, \nu - 3\omega, \frac{1}{4} - 2\nu + 4\omega, \nu - 3\omega, \omega \right].$$

For C^3 continuity, $a_3(z)$ should satisfy (15). This implies $\nu = 3\omega + \frac{1}{16}$. From this fact, we have

$$a_3 = 8 \left[\omega, \frac{1}{16}, \frac{1}{8} - 2\omega, \frac{1}{16}, \omega \right],$$

and if

$$\left\| \frac{1}{2} S_3 \right\|_{\infty} = \max \left\{ 8|\omega| + 4 \left| \frac{1}{8} - 2\omega \right|, \frac{1}{2} \right\} < 1,$$

then this scheme is $C^2(R)$.

We get the mask of S_4

$$a_4 = 16 \left[\omega, \frac{1}{16} - \omega, \frac{1}{16} - \omega, \omega \right],$$

and

$$\left\| \frac{1}{2} S_4 \right\|_{\infty} = \max \left\{ 8|\omega| + 8 \left| \frac{1}{16} - \omega \right| \right\} < 1,$$

which implies that $-\frac{1}{32} < \omega < \frac{3}{32}$. Hence this scheme is $C^3(R)$ in case $\nu = 3\omega + \frac{1}{16}$ and $-\frac{1}{32} < \omega < \frac{3}{32}$. We can see that $a_4(z)$ satisfy (15) for C^4 continuity. From the mask of S_4 , we have the mask of scheme S_5

$$a_5 = 32 \left[\omega, \frac{1}{16} - 2\omega, \omega \right].$$

From the necessary condition for C^5 continuity, we get $\omega = \frac{1}{64}$ and $\nu = \frac{7}{64}$ and

$$a_5 = \left[\frac{1}{2}, 1, \frac{1}{2} \right].$$

Since

$$\left\| \frac{1}{2} S_5 \right\|_{\infty} = \max \left\{ \frac{1}{2}, \frac{1}{2} \right\} < 1,$$

this scheme is $C^4(R)$. We get the mask of S_6

$$a_6 = [1, 1],$$

and

$$\left\| \frac{1}{2} S_6 \right\|_{\infty} = \max \left\{ \frac{1}{2}, \frac{1}{2} \right\} < 1.$$

Hence this scheme is $C^5(R)$.

| | 4-point approximating scheme | proposed scheme |
|--------------------|-------------------------------------|------------------------|
| Support | $[-4, 3]$ | $[-4, 3]$ |
| Maximal regularity | C^2 | C^5 |

Table 1. Comparison of 4-point Dyn approximating scheme and proposed scheme

In Table 1, we compare support and maximal smoothness of 4-point Dyn scheme with those of the proposed scheme. We can see that for a given same support of limit function, the proposed scheme provides good smoothness.

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