Rational Ruled surfaces construction by interpolating dual unit vectors representing lines

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ABSTRACT

In this paper, a new representational model is introduced for the rational family of ruled surfaces in Computer Graphics. The surface parameterization is constructed using the NURBS basis functions and line geometry. The ruled surface is defined by interpolating directly dual unit vectors representing lines, which is a single parametric surface and its shape depends on the control lines. All the advantages of the NURBS basis such as shape control and the local modification property are also applicable and bequeathed to the dual NURBS ruled surface.

The problem of drawing the lines defined by dual unit vectors is also resolved. Towards this direction, we propose a simple technique to calculate the surface's striction curve in order to draw the rulings of the surface within the striction curve neighborhood.

The on-screen 3D plot of the surface is realized in a pre-defined specific region close to the striction curve. With the proposed technique a natural representation of the ruled surface is derived. The shape of the surface can be intrinsically manipulated via the control lines that possess one more degree of freedom than the control points. Our method can find application not only in CAD but in the areas of NC milling and EDM.

Keywords
Ruled surface, dual unit vectors, Line Geometry, Plücker coordinates, Striction curve.

1. INTRODUCTION

A 3D surface is called ruled if through any of its points passes at least one line that lies entirely on that surface. The generation of these surfaces is considered simple since they are created by moving a line in 3D space [Far97]. Applications of these surfaces can be found in Computer Aided Design (CAD), CAGD, NC milling, and wire Electric Discharge Machining (EDM) [Yan96]. A ruled surface can be generated by linear interpolation between two given bound curves or using tensor product surfaces [Gra97]. Alternatively, a ruled surface can be defined using a line geometry representation. Ravani and Wang [Rav91] were the first to show that this approach has the advantage of curve type algorithms and they constructed ruled surfaces of degree $3m$ from $m+1$ control lines. The idea of utilizing dual vector calculus and screw theory in CAGD is not new. Chen and Pottman [Che99], presented a method to derive an approximation of ruled surface in 4-D space by tensor product B-Spline representation. Ding [Din02], used the De Boor algorithm and screw theory to determine ruled surfaces in 6-D space. Xia [Xia00] presented an approach for the problem of motion cutter planning for side milling of ruled surfaces.

In this paper, a method is introduced to define a ruled surface by extending NURBS to dual vector calculus. Most of the published approaches deal with the definition of a ruled surface using approximation algorithms such as the De Boor and De Casteljau. In this paper the ruled surface is represented by a dual NURBS structure. The representation uses the same basis function for curve definition, where the control points are replaced by control lines described by dual unit vectors. Since the rulings of the surface are represented by dual unit vectors, the main representation issue is how to define a specific part of the entire surface. In this paper we are using the

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locus of the surface's striction points in order to represent the rulings within the neighborhood of this locus. The proposed calculation of the striction curve is focused on the properties of dual line representation and is quite simple. The rulings are projected on the screen ultimately by keeping them under a specific pixel range from the striction curve. In this way we avoid the extra vectors needed for the computer graphics representation of the dually defined rulings.

2. BACKGROUND

Dual unit vector representation of a line in space.

A line in $\Re^3$ can be represented by a dual unit vector, while the elements of this vector are known as Plücker coordinates. Dual vectors are based on the theory of dual numbers invented by Clifford [Cli73]. A dual unit vector $\hat{\underset{\sim}{L}}$ is defined as:

$$\hat{\underset{\sim}{L}} = \underset{\sim}{L} + \varepsilon \cdot \underset{\sim}{L}^0$$

where $\underset{\sim}{L}$ is the principal vector, $\underset{\sim}{L}^0$ the principal moment and ε the dual unit. The reader who is unfamiliar with dual numbers and vectors is referred to the Appendix. The vector $\underset{\sim}{L} = (L, M, N)$ defines the direction of the given line and the vector $\underset{\sim}{L}^0 = (L^0, M^0, N^0)$ is the moment of the line about the origin. The dual numbers

$$L + \varepsilon \cdot L^0, M + \varepsilon \cdot M^0, N + \varepsilon \cdot N^0$$

are the dual direction cosines of the line which are the components of the dual unit vector $\hat{\underset{\sim}{L}}$. The definition of an arbitrary line in space is shown in Fig. 1 where $\underset{\sim}{r}_p$ is the position vector of a point *p* on the line. Since $\underset{\sim}{L}^0$ is the moment of the line around the origin we have $\underset{\sim}{L}^0 = \underset{\sim}{r}_p \times \underset{\sim}{L}$ and obviously $\underset{\sim}{L} \cdot \underset{\sim}{L}^0 = \underset{\sim}{L}^0 \cdot \underset{\sim}{L} = (\underset{\sim}{r}_p \times \underset{\sim}{L}) \cdot \underset{\sim}{L} = 0$. The symbol $\cdot$ declares the dot product and the symbol $\times$ the cross product. The vectors $\underset{\sim}{L}$ and $\underset{\sim}{L}^0$ are always orthogonal. Additionally, L, M, N are chosen such that $\underset{\sim}{L} \cdot \underset{\sim}{L} = L^2 + M^2 + N^2 = 1$. Since $\underset{\sim}{L} \cdot \underset{\sim}{L}^0 = 0$ and $\underset{\sim}{L} \cdot \underset{\sim}{L} = 1$ these equations are imposed to the six Plücker coordinates resulting to only four independent variables of degrees of freedom for the definition of a line in 3D space.

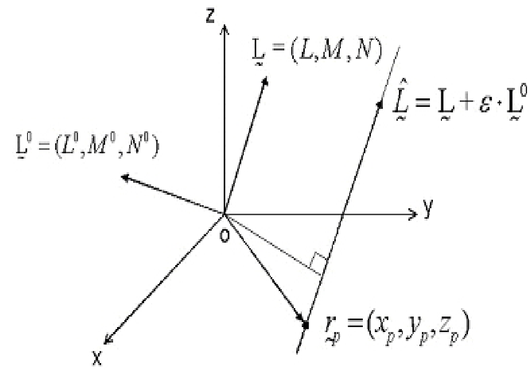

Figure 1: The Plücker coordinates of a line in the space.

Thus, the three dimensional space can be considered to be composed of ∞^4 lines instead of ∞^3 points. A line has four degrees of freedom while a point has only three but in some cases it is convenient to treat a 3D space as a set of lines and to represent its properties in terms of these lines [Roo78].

3. NURBS

NURBS curves

In this part of the paper we briefly outline the properties of NURBS. NURBS curves are rational and present some nice properties such as the ability to represent conic sections and free form surfaces on a rich geometric domain. NURBS also present the properties of the B-Splines, such as the strong convex hull and the local modification property. In fact, a NURBS curve is transformed to a B-Spline curve if all its weights are set equal to 1. The weight which is the additional parameter for the definition of a NURBS curve is quite advantageous. This extension provides one more degree of freedom for shape design and therefore makes NURBS curves more powerful than B-Splines. A NURBS curve is defined by the equation:

$$C(u) = \frac{1}{\sum_{i=0}^{n} N_{i,p}(u) w_i} \sum_{i=0}^{n} N_{i,p}(u) w_i P_i = \sum_{i=0}^{n} R_{i,p}(u) P_i \quad \textbf{(1)}$$

where $N_{i,p}(u)$ is the *i*-th basis function of degree *p* defined by:

$$N_{i,0}(u) = \begin{cases} 1 & \textit{if } u \in [u_i, u_{i+1}) \\ 0 & \textit{otherwise} \end{cases}$$

$$N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) +$$

$$+ \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+p}} N_{i+1,p-1}(u)$$

where

$$R_{i,p}(u) = \frac{N_{i,p}(u) w_i}{\sum_{j=0}^{n} N_{j,p}(u) w_j}, \quad 0 \le i \le n \quad \textbf{(2)}$$

are the NURBS basis functions and $w_i \ge 0$ the weight to control point P_i. For the NURBS curve we have: **a)** n+1 control points P_i $(0 \le i \le n)$, **b)** a knot vector U that holds $m+1$ knots and $0 = u_0 \le u_1 \le u_2 ... \le u_{m-1} \le u_m = 1$, **c)** a degree p satisfying $m = n + p + 1$. NURBS curves are rational since $R_{i,p}(u)$ are rational functions.

Dual definition of NURBS ruled surface

The dual definition of the NURBS ruled surface is straight-forward. The control points are replaced by dually defined lines using the corresponding dual unit vector $\hat{\underset{\sim}{L}} = \underset{\sim}{L} + \varepsilon \cdot \underset{\sim}{L}^0$. The equation:

$$\hat{\underset{\sim}{M}}(u) = \sum_{i=0}^{n} R_{i,p}(u) \cdot \hat{\underset{\sim}{L}}_i \quad \textbf{(3)}$$

defines a NURBS ruled surface where $\hat{\underset{\sim}{L}}_i$ are the dual unit vectors representing the control lines and $R_{i,p}(u)$ are the NURBS basis functions given by Eq (2). For the dual definition of the NURBS ruled surface we have the following: **a)** n+1 control lines represented by the corresponding dual unit vector $\hat{\underset{\sim}{L}}_i$, $(0 \le i \le n)$, **b)** a knot vector U that holds $m+1$ knots and $0 = u_0 \le u_1 \le u_2 ... \le u_{m-1} \le u_m = 1$, **c)** a degree p satisfying $m = n + p + 1$. In order to fully understand the meaning of the weight, let us consider a control line $\hat{\underset{\sim}{L}}_i$ and a weight $w_i \ge 0$. The multiplication of the control line by the weight $w_i \ge 0$ gives: $w_i \cdot \hat{\underset{\sim}{L}}_i = w_i \cdot \underset{\sim}{L}_i + \varepsilon \cdot (w_i \cdot \underset{\sim}{L}^0_i)$ which is the same line as this operation has no affect to its position and to its direction due to the use of unit vectors. However, it affects the shape of the ruled surface since we can assign a specific weight on one or more control lines. In Fig. 2, the same ruled surface is projected but with the use of weights $w_2 = 1.1$ and $w_2 = 1.4$ assigned to the control line represented by $\hat{\underset{\sim}{L}}_2$. The result is apparent; the ruled surface has been "pulled" from the corresponding control line, maintaining local control at the same time. For reasons of simplicity, in the rest of this paper, all examples are constructed utilizing weights equal to one. Moreover, for illustration purposes, throughout this paper the control lines are colored red and the striction curve in bold blue.

In Fig. 3 and 4 two dually defined NURBS ruled surfaces with the control hull and their striction curve are shown defined by 5 and 4 control lines respectively. In Fig. 3 and 4, it can be noticed that the first and last control lines, coincide with the rulings of the surface. In both these examples, the boundaries of the control hull are marked with a dashed red line. From these illustrations it is evident that the dually defined ruled surface inherits two of the most important properties that NURBS curves possess: The local effect of weights to the shape of the surface and the "restriction" of the surface by the hull that the control lines form.

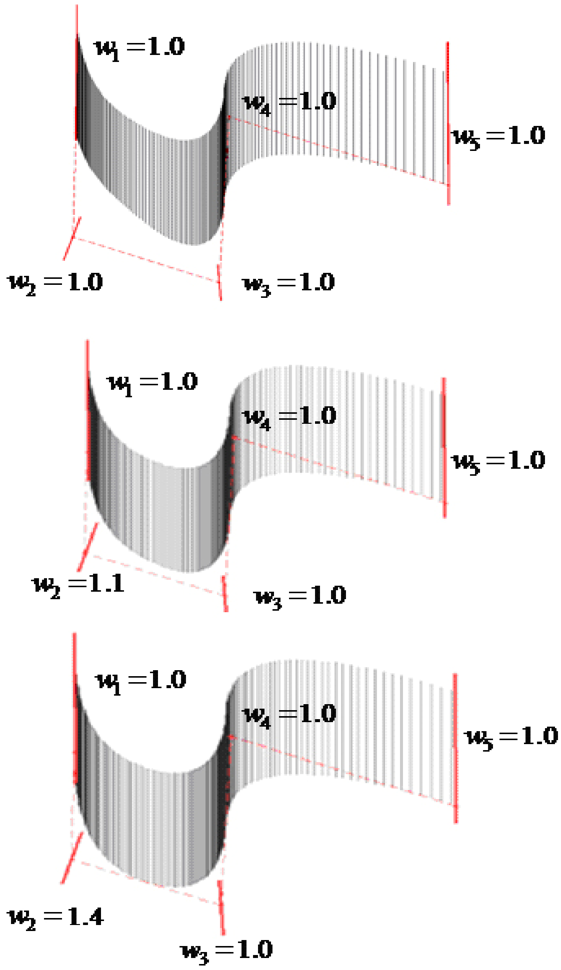

Figure 2 Ruled Surface with alternative weights ($w_2 = 1.1$ in the second case and $w_2 = 1.4$ in the third)

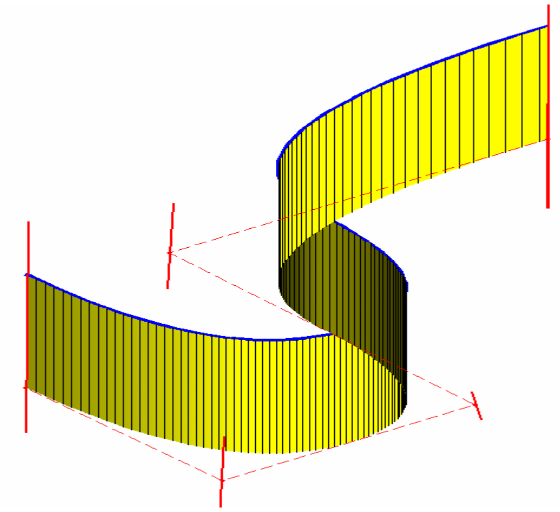

Figure 3 Control hull of 5 dually defined control lines and striction curve.

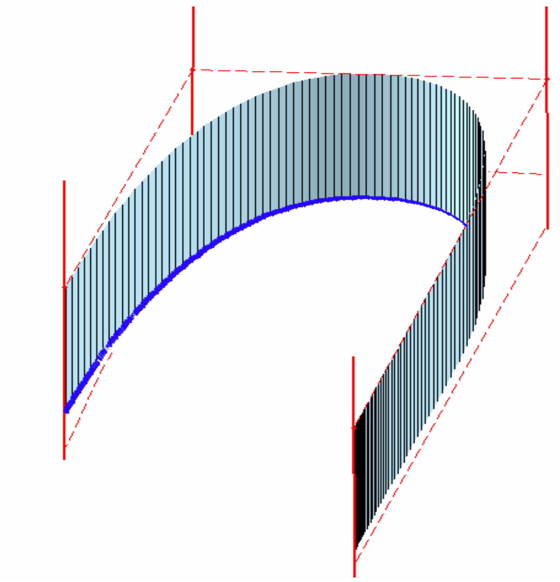

Figure 4 Control hull of 4 dually defined control lines and striction curve.

Since NURBS curves can represent conic sections and a circle, it is clear that using Eq.(3) representations of conical surfaces, generalized cylinders etc can be derived easily. For a specific value of u, Eq.(3) returns a dual line vector:

$$\hat{\underset{\sim}{M}}(u) = \underset{\sim}{M}(u) + \varepsilon \cdot \underset{\sim}{M}^0(u)$$

We normalize the $\underset{\sim}{M}(u)$ vector to obtain the unit vector:

$$\underset{\sim}{S}(u) = \frac{\underset{\sim}{M}(u)}{\left|\underset{\sim}{M}(u)\right|}$$

which is the direction of the line. Then we subtract the product of the pitch $p(u)$ and $\underset{\sim}{M}(u)$ from $\underset{\sim}{M}^0(u)$:

$$\underset{\sim}{S}^0(u) = \underset{\sim}{M}^0(u) - p(u) \cdot \underset{\sim}{M}(u)$$

where $p(u)$ is given by:

$$p(u) = \frac{\underset{\sim}{M}(u) \cdot \underset{\sim}{M}^0(u)}{\underset{\sim}{M}(u) \cdot \underset{\sim}{M}(u)}.$$

The derived dual unit vector:

$$\hat{\underset{\sim}{S}}(u) = \underset{\sim}{S}(u) + \varepsilon \cdot \underset{\sim}{S}^0(u)$$

satisfies the Plücker conditions and represents a ruling of the ruled surface for a given value of u. The main problem with this kind of representation is that line segments cannot be easily represented and therefore line to point transformations are needed [Rav91]. In this paper we introduce another approach for the representation of the rulings by utilizing the striction curve of the surface. The striction curve plays an important role in the design and in the differential geometry of ruled surfaces since it possesses some interesting properties. The maximum Gaussian curvature of the generators is located on each striction point of the surface. The point on the ruling which is closest to the successive ruling is called the striction point and is reference system independent. The locus of the striction points, known as striction curve, presents interesting properties and is useful for the evaluation of ruled surfaces in general. A non-cylindrical ruled surface can be re-parameterized using a striction-director curve representation taking into account that the striction curve does not depend on the choice of the base curve. Furthermore, the striction curve is always in contact with all the rulings of the surface and all the singularities of the surface are located on this curve.

4. DETERMINATION OF THE STRICTION CURVE

The locus of the striction points is called striction curve and has attracted the interest of researchers due to the role that plays in differential geometry of ruled surfaces. Pottman [Pot96] calculated the striction curve via the De Casteljau algorithm and line geometry. The authors of [Sch98], presented a technique for the calculation of the striction curve of a ruled surface utilizing its classical definition and dual numbers. The definition was derived utilizing the arc length of the indicatrix curve as the parameter and was depended on an arbitrary directrix of the ruled surface. In this paper, the striction curve is calculated using elements that are included in the proposed line representation of the ruled surface.The striction points are determined as follows: Let $\hat{\underset{\sim}{N}}(u, \Delta u)$ be the common perpendicular between two successive rulings $\hat{\underset{\sim}{S}}(u)$ and $\hat{\underset{\sim}{S}}(u + \Delta u)$, $u \in \Re$, where $\Delta u \in \Re$ is very small as it is shown in Fig. 5.

Let

$$\hat{\underset{\sim}{S}}(u) = \underset{\sim}{S}(u) + \varepsilon \cdot \underset{\sim}{S}^{0}(u)$$

and

$$\hat{\underset{\sim}{S}}(u+\Delta u) = \underset{\sim}{S}(u+\Delta u) + \varepsilon \cdot \underset{\sim}{S}^{0}(u+\Delta u)\,.$$

Then the common perpendicular is given by:

$$\begin{aligned}&\hat{\underset{\sim}{N}}(u,\Delta u) = \underset{\sim}{S}(u) \times \underset{\sim}{S}(u+\Delta u) + \\ &\varepsilon \cdot (\underset{\sim}{S}(u) \times \underset{\sim}{S}^{0}(u+\Delta u) + \underset{\sim}{S}^{0}(u) \times \underset{\sim}{S}(u+\Delta u))\end{aligned} \qquad \textbf{(4)}$$

in dual vector form. The reader who is not familiar with the dual number operations is referred to the appendix. Utilizing the straight forward definition of the dual point [Aza01] by two dual unit vectors we get for the striction point $S_p(u)$:

$$\begin{aligned}&S_p(u) = \underset{\sim}{S}(u) \times \underset{\sim}{S}(u+\Delta u) + \\ &(\underset{\sim}{N}(u,\Delta u) \times \underset{\sim}{N}^{0}(u,\Delta u) \cdot \underset{\sim}{S}(u)) \cdot \underset{\sim}{S}(u)\end{aligned} \qquad \textbf{(5)}$$

where

$$\hat{\underset{\sim}{N}}(u,\Delta u) = \underset{\sim}{S}(u) \times \underset{\sim}{S}(u+\Delta u)$$

and

$$\underset{\sim}{N}^{0}(u,\Delta u) = (\underset{\sim}{S}(u) \times \underset{\sim}{S}^{0}(u+\Delta u) + \underset{\sim}{S}^{0}(u) \times \underset{\sim}{S}(u+\Delta u))$$

Two dual unit vectors intersect when the dual component of their dual scalar product vanishes. This property is used for the above definition of the striction point. The resulting construction has enough freedom to hold three point vectors simultaneously. Thereby, extra information can be stored into the expression of the striction point such as the tangent vector, the curvature vector etc. In Fig. 5 the striction point $S_p(u)$ is illustrated for two successive rulings on a ruled surface along with the common perpendicular $\hat{\underset{\sim}{N}}(u,\Delta u)$.

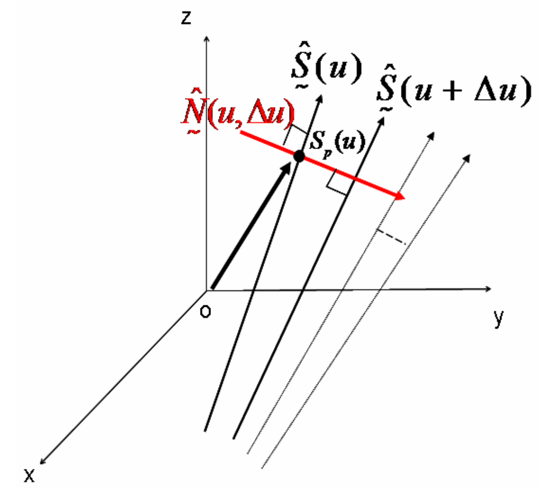

Figure 5 *The Striction Point* $S_p(u)$.

5. RULED SURFACE DRAWING

Our method is based on the calculation of the striction curve for the rendering of dual NURBS ruled surfaces. In the current literature, the problem of drawing an infinite line represented by a dual vector has been addressed via several ways. Ravani and Wang [Rav91] introduced a method that utilized the “centre point” of a line to define a line segment. They used two additional parameters on the line coordinates to do so. However, the location of the centre point cannot be controlled during the interpolation. Ding [Din02] used the dual De Boor algorithm for the on screen representation of screws. Sprott and Ravani [Spr97] proposed a method that used a point on the first control line as a reference in order to define a line through that point. Then they derive a surface construction interpolating the control screws with the reference line. These methods mainly utilize a line-segment scheme or approximation algorithms in order to “pass” from the dual number “world” to the screen. The technique proposed in this paper, is not based on an approximation algorithm such as De Boor or De Casteljau but on the exact calculation of the striction points.

The striction points always lie on the common perpendicular among successive rulings and their locus forms a curve of minimal length that meets all the rulings. Using this property, we draw each one of the rulings of the surface by plotting the straight line segments in the neighborhood of the striction curve. More precisely, we project onto the computer screen a specific number of pixels from the striction curve along a ruling. Thus, the need for the two point definition of a line segment is eliminated since the only necessary parameter is the distance from the striction curve. In the following illustrated examples, we demonstrate the difference between two projections of the same ruled surface and of alternative striction curve neighbourhoods. In Fig. 6 and 7 a dual NURBS ruled surface is shown along with the striction curve. It is obvious that in Fig. 7 only the pixels in one side of the striction curve are projected. Finally, two complete shaded models of two ruled surfaces via the dual NURBS definition are represented in Fig. 8 and 9.

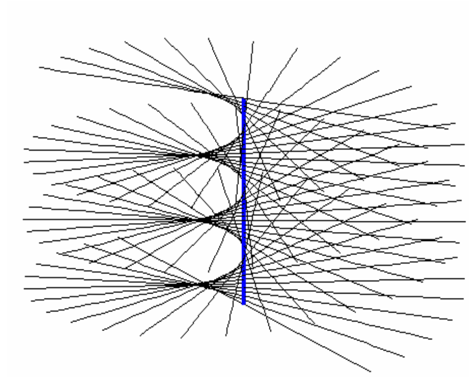

Figure 6 Dual NURBS ruled surface (500 pixels for the first ruling) and striction curve $S_p(u)$.

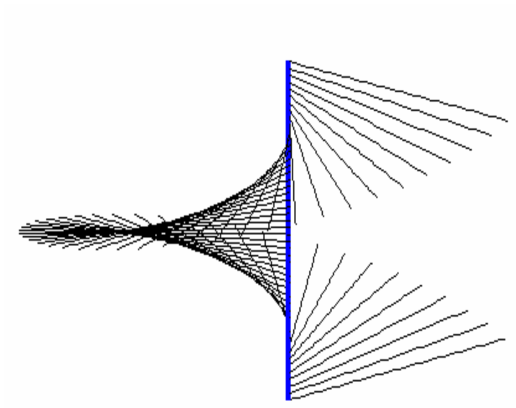

Figure 7 Dual NURBS ruled surface (250 pixels for the first ruling) and striction curve $S_p(u)$.

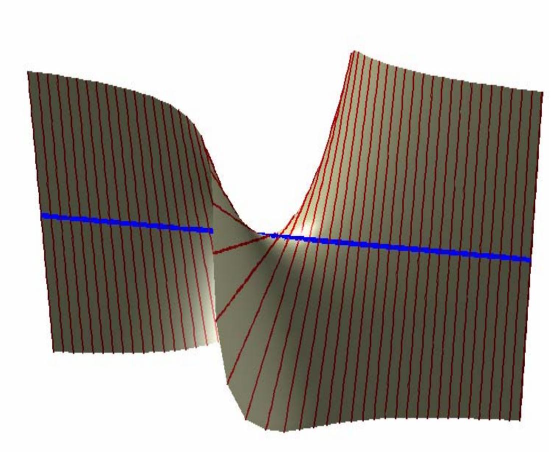

Figure 8 Dual NURBS ruled surface and striction curve $S_p(u)$.

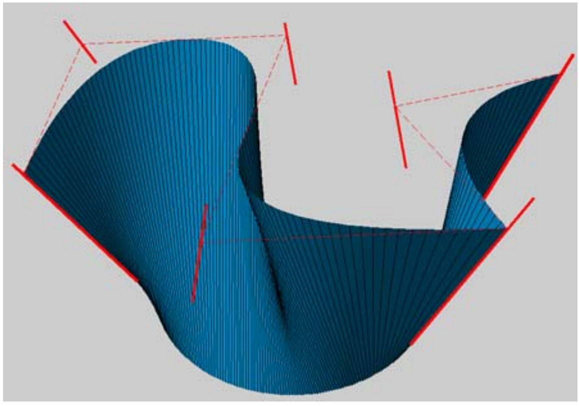

Figure 9 Dual NURBS ruled surface and control lines

6. CONCLUSIONS

In this paper, a new method for the representation of ruled surfaces using line geometry has been proposed. We have developed an alternative way for the definition of ruled surfaces utilizing dual unit vectors, the NURBS basis functions and the striction curve. The problem of projecting straight lines represented by dual unit vectors is resolved by the utilization of the striction curve. We provide a way for the calculation of the surface's striction points. The calculation is based upon the dual definition of the surface and is performed utilizing the richness of the dual vector representation. Moreover, the definition of the striction points is used to address another problem of the "dual world": the on screen projection of dual unit vectors representing lines. Our representation, can find application in the areas of CAD and CAGD.

In the last decade, line geometry and dual vector calculus has attracted the interest of many researchers and has been studied from the design point of view. However, point-based representations are still dominating CAD and CAGD. This is due to the fact that modern computer graphics systems can only produce 3D to 2D projections point wise. Future research should include the utilization of line geometry by modern APIs. Moreover, the study of the transformations of these surfaces presents great interest. Dual quaternions and dual matrices that have been proved to be smooth transformation operators of dual vectors, are two natural candidates for future investigation.

7. REFERENCES

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APPENDIX

Dual numbers can be defined as:

$\hat{a} = a + \varepsilon \cdot a^0, \; a, a^0 \in \Re$

where ε is the dual unit with the properties of:

$\varepsilon \neq 0, \; 0 \cdot \varepsilon = 0, \; 1 \cdot \varepsilon = \varepsilon \cdot 1 = \varepsilon, \; \varepsilon^2 = 0$

The multiplication and addition operations of two dual numbers $\hat{a}_1 = a_1 + \varepsilon \cdot a_1^0, \; \hat{a}_2 = a_2 + \varepsilon \cdot a_2^0$ are defined as:

Addition: $\hat{a}_{\mathbf{1}} + \hat{a}_2 = (a_1 + \varepsilon \cdot a_1^0) + (a_2 + \varepsilon \cdot a_2^0) = (a_1 + a_2) + \varepsilon \cdot (a_1^0 + a_2^0)$

Multiplication:

$\hat{a}_{\mathbf{1}} \cdot \hat{a}_2 = (a_1 + \varepsilon \cdot a_1^0) \cdot (a_2 + \varepsilon \cdot a_2^0) = a_1 \cdot a_2 + \varepsilon \cdot (a_1 \cdot a_2^0 + a_1^0 \cdot a_2)$

External product: $\underset{\sim}{\hat{a}}_1 \times \underset{\sim}{\hat{a}}_2 = \underset{\sim}{a}_1 \times a_2 + \varepsilon \cdot (\underset{\sim}{a}_1 \times \underset{\sim}{a}_2^0 + \underset{\sim}{a}_1^0 \times \underset{\sim}{a}_2)$