Dual Subdivision

A New Class of Subdivision Schemes Using Projective Duality

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ABSTRACT

This paper proposes a new class of subdivision schemes. Previous subdivision processes are described by the movement and generation of vertices, and the faces are specified indirectly as polygons defined by those vertices. In the proposed scheme, on the other hand, the subdivision process is described by the generation of faces, and the vertices are specified indirectly as the intersections of these faces. In this sense, this paper gives a framework for a wide class of new subdivision methods. In short, the new subdivision is a dual framework of an ordinary subdivision based on the principle of duality in projective geometry. So, the new subdivision scheme inherits various properties of the ordinary subdivision schemes. In this paper, we define the dual subdivision and derive its basic properties.

Keywords: subdivision surface, dual schemes, smoothness analysis

1 INTRODUCTION

Subdivision [11, 13, 3, 22, 20] is a well-known method for geometric design and for computer graphics, because the subdivision makes smooth surfaces with arbitrary topology. A subdivision scheme is defined by subdivision matrices and a rule of connectivity change. So, many researchers study the condition of continuity of subdivision surfaces depending on subdivision matrices [22, 24, 17, 16, 1, 7, 23, 18, 4]. Moreover, multiresolution analysis [14, 21, 5] derived by subdivision theory is extremely useful on mesh editing.

Subdivisions on quadrilateral or triangular meshes were studied extensively. For example, the most popular subdivisions are the Catmull-Clark subdivision and the Loop subdivision [13]. These subdivisions are designed for irregular quadrilateral or triangular meshes.

Most subdivision methods are for triangular or quadrilateral meshes; there are only a few methods for other types of meshes. Some subdivisions on hexangular meshes were developed [2, 6]. However, faces generated by these subdivisions are not “flat".

In this paper, we derive new subdivision schemes. This is the dual framework of an ordinary subdivision based on the principle of duality in projective geometry. The proposed dual subdivision can generate meshes, composed of non-triangular “flat" faces.

Approximating surfaces using non-triangular flat faces is a basic problem for computational geometry. If the surface is convex, the approximation is easy. However, if the surface is not convex, the approximation can not be completed yet [19]. This dual subdivision schemes overcome this problem.

The dual subdivision is a wide class of new subdivisions. The dual subdivision generates faces by subdivision matrices. The subdivision matrices are shared between an ordinary subdivision and the corresponding dual subdivision. So, the dual subdivision has properties similar to the ordinary subdivision. In this paper, we explain such properties.

Levin and Wartenbarg [12] already proposed some dual schemes. However, their schemes are convexity-preserving interpolations. In first place, dual subdivision can naturally represent surfaces with arbitrary topology. Our framework enables it by “inflection plane".

From the mathematical view point, in ordinary subdivisions, basis functions are attached to vertices. In dual subdivisions, basis functions are attached to faces (equations of faces). Moreover, in line subdivisions [10], basis functions are attached to edges (the Klein image of edges [15, 8]). So, dual subdivision is an important element of subdivisions.

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2 ORDINARY SUBDIVISION

In this section, we review a general subdivision.

2.1 Subdivision Matrix

A subdivision scheme is defined by subdivision matrices and a rule of connectivity change. The subdivision scheme, when it is applied to 2-manifold irregular meshes, generates smooth surfaces at the limit. Fig. 1 is an example of the Loop subdivision. In this figure, (a) is an original mesh; subdividing (a), we get (b); subdividing (b) once more, we get (c); subdividing infinite times, we get the smooth surface (d). We call (d) the subdivision surface. Here, a face is divided into four new faces. This is a change of connectivity. In this paper, the change of connectivity is fixed to this type, but other types of connectivity change can be argued similarly.

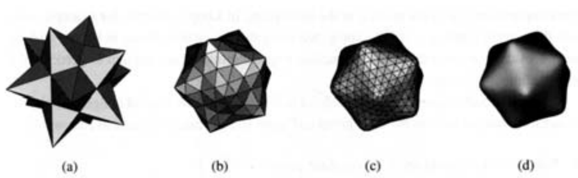

Figure 1: Loop subdivision [21].

Next, let us consider how to change the positions of the old vertices, and how to decide the positions of the new vertices. They are specified by matrices called "subdivision matrices". The subdivision matrices are defined at vertices and they depend on degree k of the vertex (the degree is the number of edges connected to the vertex). For example, Fig. 2 denotes a vertex v_0^j which has five edges. Let $v_1^j, v_2^j, \cdots, v_5^j$ be the vertices at the other terminal of the five edges. Then, subdivision matrix S_5 is defined as follows:

$$\begin{pmatrix} v_0^{j+1} \\ v_1^{j+1} \\ \vdots \\ v_5^{j+1} \end{pmatrix} = S_5^j \begin{pmatrix} v_0^{j} \\ v_1^{j} \\ \vdots \\ v_5^{j} \end{pmatrix}.$$

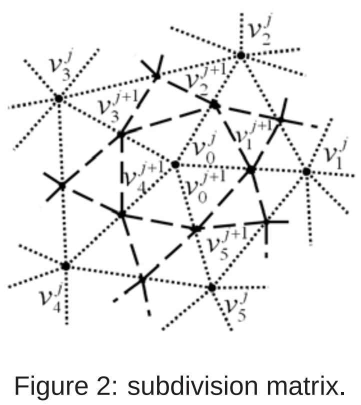

Figure 2: subdivision matrix.

Here, the subdivision matrix S_5^j is a square matrix. j means j-th step of the subdivision. Here, neighbor vertices of a vertex v are called vertices on the 1-disc of v. The subdivision matrix is generally defined not only on vertices in the 1-disc, but also on other vertices in 2-disc, 3-disc, $\cdots$. Here, we argue only subdivision matrices that affect vertices in the 1-disc. However, we can argue other subdivision matrices, similarly. In this paper, we assume that the subdivision matrix is independent on j. A subdivision scheme of this type is called "stationary".

In this way, subdivision matrix S_k is written for a vertex. However, since a newly generated vertex is computed by two subdivision matrices at the ends of the edge, the two subdivision matrices must generate the same location of the vertex. So, the subdivision matrices have this kind of restriction.

The degree k of a vertex is at least two. A vertex whose degree is two is a boundary vertex. The degree of a vertex of 2-manifold meshes is at least three. In this paper, we do not argue boundaries of meshes. So, we assume that the degree is at least three.

As seen above, a stationary subdivision scheme is defined by subdivision matrices S_k ($k \geq 3$). Then, from the theorem 2.1 in [1], the regular limit surface of subdivision $f : |K| \to \mathbf{R}^3$ is the following parametric surface:

$$f[p](y) = \sum_i v_i \phi_i(y),$$

$$v_i \in \mathbf{R}^3,\ \phi_i(y) \in \mathbf{R},\ y \in |K|, p = (v_0, v_1, \cdots)^\top,$$

where K is a complex, $|K|$ is a topological space, that is, the mesh, y is a local two-parameter, that is, locally $y = (y_1, y_2)$, i is an index of a vertex, v_i is the position of the i-th vertex, $\phi_i(y)$ is the weight function with the i-th vertex. Moreover, the weight function $\phi_i(y)$ is dependent only on the subdivision matrices. In what follows we assume that the sum of element in each row of the subdivision matrix is equal to 1. Here, the operation for generating the vertices of the $j+1$-st step of the subdivision from the vertices of the j-th step is affine invariant; it does not depend on the origin of the coordinate system.

Here, we denote $\phi(y) = (\phi_0(y), \phi_1(y), \cdots)$. Then, $\phi(y)$ decides a set of representable surfaces. Then, the set is spanned by $\phi(y)$. So, we call the weight functions basis functions. The limit surface of the subdivision is a point in such a functional space.

3 DUAL SUBDIVISION

Here, we propose a dual subdivision method.

3.1 Dual Transformation

The transformation $(p_x, p_y, p_z, p_w) \leftrightarrow p_x x + p_y y + p_z z - p_w w = 0$ is a well-known duality called projective duality in P^3. In this paper, we use $(a, b, c) \leftrightarrow ax + by + cz - 1 = 0$ which is a projective duality. We denote this duality by D, that is, for a point p, $D(p)$ represents its dual plane, and for a plane h, $D(h)$ represents its dual point. Here, the transform satisfies the following properties:

- When a point p is on a hyperplane h, and only then, the point $D(h)$ is on the hyperplane $D(p)$.
- When a point p exists in the upper (lower) half-space partitioned by a hyperplane h, the point $D(h)$ exists in the upper (lower) half-space partitioned by the hyperplane $D(p)$. Here, a lower half-space means the half-space which has the origin and the upper half-space means the other (the half-space does not contain the separating plane.).

3.2 Definition of Dual Subdivision

The ordinary subdivision is specified by how the vertices are generated and located. On the other hand, the dual subdivision, which we will define here, is specified by how the faces are generated and located.

We assumed that the sum of each row of the subdivision matrix is 1.

Here, p^j is a column vector of vertices at the j-th subdivision step. Using a subdivision matrix S, p^{j+1} is written as:

$$p^{j+1} = Sp^j,$$

where

$$p^j = \begin{pmatrix} p^j_{0x} & p^j_{0y} & p^j_{0z} \\ p^j_{1x} & p^j_{1y} & p^j_{1z} \\ & \vdots & \end{pmatrix}.$$

Therefore, if we denote

$$f^j = \begin{pmatrix} p^j_{0x} & p^j_{0y} & p^j_{0z} & -1 \\ p^j_{1x} & p^j_{1y} & p^j_{1z} & -1 \\ & \vdots & & \end{pmatrix},$$

we get

$$f^{j+1} = Sf^j,$$

where the elements of each row of f^j are coefficients of the equation $p^j_{ix}x + p^j_{iy}y + p^j_{iz}z - 1 = 0$. Therefore, the equations of planes are subdivided. These equations are the dual of vertices $(p^j_{ix}, p^j_{iy}, p^j_{iz})$. So, this subdivision is a dual framework of ordinary subdivision. Moreover, dual subdivision can be defined by a projective duality $(p_x, p_y, p_z, p_w) \leftrightarrow p_x x + p_y y + p_z z - p_w w = 0$.

Now, for any triangular mesh M, using the duality, we get a dual mesh $D(M)$. Here, if the degree of a vertex v of M is k, then v is the intersection of faces f_i, $i = 1, 2, \cdots, k$, so these vertices $D(f_i)$, $i = 1, 2, \cdots, k$ is on the face $D(v)$. So, we get following observation:

Observation 3.1 (Dual mesh)
If the degree of the vertex v of M is k, the face $D(v)$ of $D(M)$ is a k-gon.

Note that even if meshes in primal space are bounded, dual meshes are not necessarily bounded. First, clearly, dual meshes depend on the origin in primal space. A point $\alpha a + \beta b$ on a face p is the convex combination of neighbor vertices a, b. So, we can see the plane $D(\alpha a + \beta b) = \alpha D(a) + \beta D(b)$ is a tangent plane of $D(p)$. Thus we can define the tangent plane of $D(p)$ as convex combinations of tangent planes of the neighborhood (See the upper figure in Fig. 3.). If a tangent plane $\alpha p + \beta q$ of a vertex contains the origin in primal space, then the vertex $D(\alpha p + \beta q)$ is a point at infinity, and so the face, which has vertices $D(p), D(q)$, is not bounded (See the lower figure in Fig. 3.). Therefore, we want to know the condition for dual meshes and dual subdivision surfaces to be bounded. So, we derived a necessary and sufficient condition for dual meshes and a sufficient condition for dual subdivision surfaces in [9].

Here, we define the rule of connectivity change of dual subdivision. The connectivity change of dual subdivision is defined as dual of the connectivity change of ordinary subdivision (See Fig. 4).

So, we get following observation.

Observation 3.2 (Dual subdivision)
Applying ordinary subdivision to meshes in the primal space is dual of applying dual subdivision to dual meshes in the dual space.

Here, we can see that if meshes made by ordinary subdivision approximate surfaces very well, dual meshes made by dual subdivision approximate dual surfaces very well, too. Thus, dual subdivision has a useful property that it can represent surfaces by "flat" polygons on non-triangular meshes, for example, hexagonal meshes.

3.3 Properties of Duality

In this subsection, we explain an important property. The property is based on dual transformation $(a, b, c) \leftrightarrow ax + by + cz - 1 = 0$ and 2-manifold.

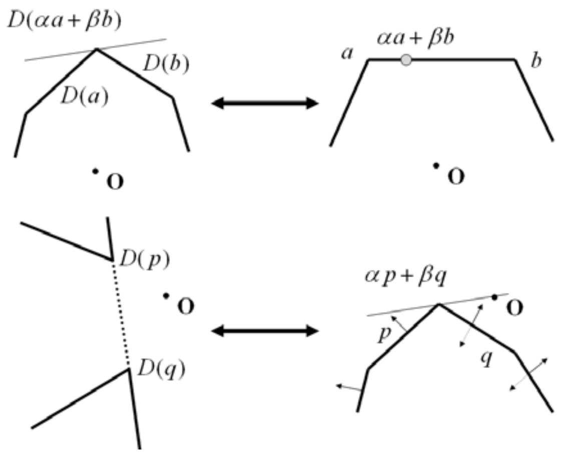

Figure 3: Tangent planes of a vertex. In the upper figure, we can see the tangent plane of a vertex as convex combinations of tangent planes of the neighborhood. So, if a tangent plane $\alpha p + \beta q$ contains the origin, $D(\alpha p + \beta q)$ is a point at infinity. Therefore, the face is not bounded. Here, q is a plane $-p_x x - p_y y - p_z z + 1 = 0$ whose normal is $(-p_x, -p_y, -p_z)$. If we write the equation of q as $p_x x + p_y y + p_z z - 1 = 0$, then the normal is (p_x, p_y, p_z) (this normal is denoted by the dashed arrow). We consider that meshes have continuous normals, that is, meshes are oriented. Thus, we define the equation of q as $-p_x x - p_y y - p_z z + 1 = 0$. Then, the un-bounded face is convex combinations of $D(p)$ and $D(q)$, too.

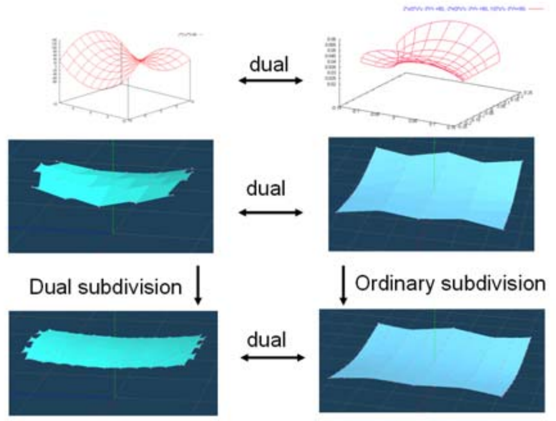

Figure 4: The duality between ordinary subdivision and dual subdivision. Upper left drawing is a surface which has saddle points. Upper right drawing is the dual surface. Middle right picture is a triangular mesh made by plotting points on upper right surface. Middle left mesh is the dual mesh of the middle right mesh. we get lower right mesh made by ordinary subdivision for the middle right mesh. On the other hand, we get lower left mesh made by dual subdivision for the middle left mesh. Then, the lower left mesh is the dual mesh of the lower right mesh. Dual subdivision is defined as such. Like this, dual subdivision can represent surfaces which have saddle points.

So, the property holds independently of the dual subdivision.

Here, we denote a regular 2-manifold in the primal space as S, the dual shape of S as $D(S)$. Points of $D(S)$ are the dual of tangent planes of S. Tangent planes of $D(S)$ is the dual of points of S. We can see $D(S)$ as an envelope surface defined by the dual of points of S. Here, "flat" means that the points of the subset of a surface share a tangent plane.

In this paper, we assume that a 2-manifold is connected and has a finite genus.

First, we get following proposition.

Proposition 3.1 (Duality of 2-manifold)
If S is a bounded and regular 2-manifold, and if any subset of S and $D(S)$ are not flat, then $D(S)$ is a 2-manifold.

Proof *First, any bounded and regular 2-manifold S in the primal space has an open covering which is composed of a finite number of open sets whose topology is equal to that of a disc. Moreover, this dual transform is a continuous and one-to-one mapping. So, the open sets are mapped to open sets of tangent planes in the dual space. Since any subset of S is not flat, an envelope surface made by an open set of tangent planes is an open set of points in the dual space. Therefore, the dual surface of the open covering is represented by a union of open sets. Since the open covering is composed of a finite number of open sets, the dual surface of the open covering is an open set. Moreover, there is an open covering whose union is equal to S. So, the dual surface of this open covering is equal to $D(S)$. Therefore, there are open sets, whose topology is equal to that of a disc, at any point of $D(S)$. So, $D(S)$ is a 2-manifold.* □

$D(S)$ is generally not a 2-manifold even if S is a 2-manifold. For example, if S is a plane, then $D(S)$ is a point, because a subset of S is flat. For example, a plane is totally flat, a face of a mesh is flat, the top and bottom circle of a torus are flat. So, flat parts make degenerations for dual shapes. If the degeneration does not break the structure of 2-manifold of dual surfaces, we can easily make the degenerated surface. Otherwise, we must use special subdivision matrices and a rule of connectivity change to represent the degenerated surface.

Therefore, in this paper, we discuss non-degenerate surfaces.

4 INFLECTION PLANE

4.1 Inflection Point

We will discuss the smoothness of the limit surfaces of dual subdivision and show the duality of smoothness $C^1_{\mathbf{ordinary}} \Leftrightarrow C^1_{\mathbf{dual}}$ in P^3. However, to show the relation, we need a condition. In this section, we talk about the condition.

Even if a dual surface in the dual space is smooth and if any subset of the surface is not flat, the associated surface in the primal space is not necessarily smooth. (However, the continuity of tangent planes is guaranteed.) For example, see Fig. 5.

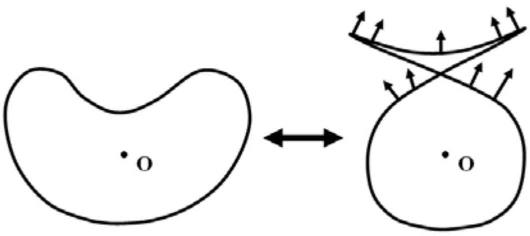

Figure 5: The left object in 2D has inflection points. The right object, which is the dual of the left object, has reversals of the normal at dual inflection points. In 3D, such thing happens, too.

Although the dual curve in the dual space is smooth, we see that the corresponding curve is not smooth in the primal space. This arises from reversals of normals at the dual of inflection points.

Here, we define an “inflection point" of surfaces in $\mathbf{R}^3$. We call a point p an “inflection point" if a cross-sectional curve of the surface at p (the cross-sectional curve is the intersection of the surface and a plane through p and the origin.) has the inflection point p.

Thus, if the C^1 dual surface has inflection points, the associated surface in the primal space is not C^1-continuous.

Consider the neighborhood of a point. We can classify the neighborhood to convex, concave, inflection. As seen above, inflection parts break the smoothness. On the other hand, the others do not break it.

4.2 Inflection Plane

To overcome this problem, we use a technique named “inflection plane".

First, we get a mesh in the primal space as Fig. 6. Applying smooth subdivision scheme to this mesh, the limit curve in primal space is smooth. So, the limit curve is not equal to the target shape in Fig. 6. Therefore, dual of the limit curve of this mesh is not equal to the left shape in Fig. 5.

Even if the original mesh in the primal space approximates the right shape very well, the limit curve in the primal space does not have reversals of normals. So, the curve is not equal to the target shape.

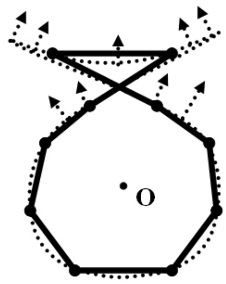

Figure 6: A mesh in primal space.

So, we want to generate the surface with reversals of normals by subdivision in the primal space.

Here, we define an “inflection plane". We add two-ply faces $AB, A'B, \cdots$ as shown in Fig. 7. These added faces generate the reversals of normals, because the basis function at point B is the delta function (Here, we assume all supports of basis functions are 1-disc. Then, the limit curve of the mesh to which $AB, A'B$ is added is tangent plane continuous.).

Like this, the two-ply face $AB, A'B$ generates the dual of the tangent plane at an inflection point in the dual space. Moreover, the tangent plane at B is the dual of an inflection point in the dual space. So, we call the two-ply face $AB, A'B$ an “inflection plane".

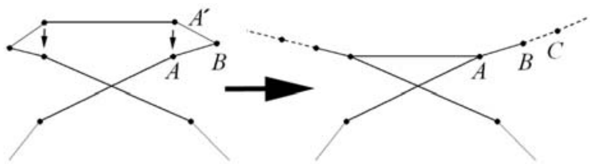

Figure 7: Adding an inflection plane $AB, A'B$. We conform the position of vertex A' to that of vertex A. So, both meshes have the same topology. We call the two-ply face $AB, A'B$ “inflection plane".

If all supports of those are over 2-disc, we add points $C, D, \cdots$. Then, we get the limit surfaces with tangent plane continuity and reversals of normals.

Similarly, using inflection planes, we can get smooth surfaces which have inflection points in dual space (see Fig. 8).

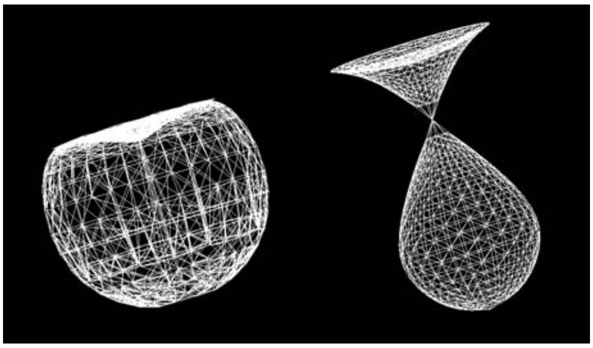

Figure 8: The left mesh which is made by dual subdivision is the dual of the right mesh and not convex and has inflection points. The right mesh which is made by ordinary subdivision using inflection plane has dual inflection points.

5 DUALITY OF SMOOTHNESS

In this section, we derive the relation $C^1_{\mathbf{ordinary}} \Leftrightarrow C^1_{\mathbf{dual}}$ in P^3. Then, we can get smooth dual limit surfaces.

Proposition 5.1 (Duality of smoothness) *Assume that S is a bounded and regular 2-manifold and any subset of S and $D(S)$ are not flat. Then $D(S)$ is tangent plane continuous if and only if S is tangent plane continuous. Moreover, if S and $D(S)$ have no inflection points, then $D(S)$ being C^1-continuous is equivalent to S being C^1-continuous.*

Proof *Since S is tangent plane continuous, points of $D(S)$ are continuous. Moreover, any subset of S is not flat. So, the dual of tangent planes of S is non-degenerate. Here, points of S are continuous. So, tangent planes of $D(S)$ are continuous. Therefore, $D(S)$ is tangent plane continuous. Similarly, since $D(S)$ is tangent plane continuous, points of S are continuous. Moreover, any subset of $D(S)$ is not flat. So, the dual of tangent planes of $D(S)$ is non-degenerate. Here, points of $D(S)$ are continuous. So, tangent planes of S are continuous. Therefore, S is tangent plane continuous.*

If S and $D(S)$ do not have inflection point, then $D(S)$ is C^1-continuous if S is C^1-continuous. Because, locally the neighborhood of a point of S is convex or concave. So, the neighborhood does not reverse normals of $D(S)$. □

To get smooth $D(S)$ which has inflection points, we must use the inflection plane.

5.1 Properties of Dual Subdivision

Next, we derive an important theorem for smoothness of limit surfaces.

Theorem 5.1 (Duality of smoothness) *Assume that there are local parameterizations, which have Jacobi matrix of maximal rank 2 except at extraordinary points, on basis functions of ordinary stationary subdivision, and there are unique tangent planes at extraordinary points, and any subset of the limit surface is not flat, and the limit surfaces of ordinary and dual subdivision have no inflection points. Then, the limit surfaces of the dual subdivision are C^1-continuous if and only if the limit surfaces of the ordinary subdivision are C^1-continuous:*

$$C^1_{\mathbf{ordinary}} \Leftrightarrow C^1_{\mathbf{dual}}.$$

Proof *For any basis function generated by ordinary stationary subdivision, if Jacobi matrix is degenerate at a point on the basis function except the extraordinary point, then Jacobi matrix is degenerate at any point of the basis function. Then, Jacobi matrix is degenerate on the part, which corresponds to the basis function, of the subdivision surface except a finite number of extraordinary points. So, Jacobi matrix of maximal rank 2 except at extraordinary points means that Jacobi matrix is non-degenerate at any point of the surface generated by ordinary subdivision except extraordinary points. Therefore there are unique tangent planes of the limit surface of ordinary subdivision except extraordinary points. Here, there are unique tangent planes at extraordinary points. So, there are unique tangent planes of the surface generated by ordinary subdivision. Thus, any subset of $D(S)$ is not flat. Moreover, any subset of the limit surface is not flat. Therefore, by proposition 5.1, we get this theorem.* □

In this way, we can guarantee the smoothness of dual limit subdivision. So, we can use the dual subdivision scheme for applications, e.g. approximating shapes.

6 CONCLUSION

In this paper, we proposed a new subdivision method. This is a dual framework of ordinary subdivision based on the projective duality. Because of the duality, dual subdivision has useful properties similar to ordinary subdivision.

First, we derived the duality of 2-manifold. Thus, we can see that the dual limit surface is 2-manifold.

Second, we defined an "inflection plane". Using the inflection plane, we can represent smooth surfaces with inflection points by dual subdivision.

Finally, we derived the relation $C^1_{\mathbf{ordinary}} \Leftrightarrow C^1_{\mathbf{dual}}$. This duality of smoothness enables us to represent smooth surfaces by "flat" non-trigonal polygons.

Moreover, we can lead multiresolution analysis [21] for dual subdivision. It is useful for the mesh editing, watermarking, etc..

For higher-order smoothness, refer to our technical report [9]. In the technical report, we derived conditions for the limit surface of dual subdivision surfaces to be C^k-continuous. Using "universal surface" [24], we derived relations of smoothness between ordinary subdivision and dual subdivision.

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