FROM DUPIN CYCLIDES TO SCALED CYCLIDES

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ABSTRACT

Dupin cyclides are algebraic surfaces introduced for the first time in 1822 by the French mathematician Pierre-Charles Dupin. They have a low algebraic degree and have been proposed as a solution to a variety of geometric modeling problems. The circular curvature line's property facilitates the construction of the cyclide (or the portion of a cyclide) that blends two circular quadric primitives. In this context of blending, the only drawback of cyclides is that they are not suitable for the blending of elliptic quadric primitives. This problem requires the use of non circular curvature blending surfaces.

In this paper, we present another formulation of cyclides: Scaled cyclides. A scaled cyclide is the image of a Dupin cyclide under an affine scaling application. These surfaces are well suited for the blending of elliptic quadrics primitives since they have elliptical lines of curvature. We also show how one can convert a scaled cyclide into a set of rational quadric Bézier patches.

Keywords: *Geometric modeling, Quadric primitives, Dupin cyclides surfaces, Supercyclides, Bézier surfaces, Blending.*

1 INTRODUCTION

Dupin cyclides are non-spherical algebraic surfaces discovered by the French mathematician Pierre-Charles Dupin at the beginning of the 19th century [Dup22]. These surfaces have an essential property: all their curvature lines are circular.

Dupin cyclides gained a lot of attention, and their algebraic and geometric properties have been deeply studied by many authors [BG92, Ber78, Bla29, Heb81, Pin85, Ban70, DMP93, Pra90, Mar82]. A Dupin cyclide is an algebraic surface of low degree (usually less than 4) having an easy to understand parametric representation (with 3 parameters) as well as two equivalent implicit equations [Pra90, For12]. In several publications several authors has studied the use of these surfaces in the blending operations, Pratt and Boehm [Boe90, Pra90], Chandru et al. [CDH89], Srinivas et al. [SKD96], Shene [She98, She00] A low algebraic degree equation and a simple parametric representation give cyclides advantages of both implicit and parametric formulations. The parametric form allows easily editing and quick visualization of these surfaces. Another important point that marks the beauty of Dupin cyclides is their well established conversions to other parametric surfaces such as Bézier, Bspline, or NURBS [Pra90, Boe90]This conversion of Dupin cyclides into rational quadric Bézier patches bridges the gap between these two modeling tools and may accelerate the introduction of cyclides into geometric modeling systems based on classical parametric surfaces.

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The circular curvature line's property facilitates the construction of the cyclide (or the portion of a cyclide) for the blending of two circular quadrics primitives [CDH89, AD97a, AD97b, SKD96, She00]. In the context of blending, the only drawback of Dupin cyclides is that they are not suitable for the blending of elliptic (non-circular) quadric primitives. This problem requires non-circular curvature blending surfaces.

Several authors have studied the generalization of Dupin cyclides to define cyclides with elliptic lines of curvature [Deg82, Deg94b, Deg94a]. Later in 1997, M. Pratt analyzed these surfaces differently and called them supercyclides, formulating them as projective transforms of Dupin cyclides [Pra97, Deg98].

In this paper, we present a special case of supercyclides : Scaled cyclides. A scaled cyclide is the image of a Dupin cyclide under an affine scaling application. These surfaces are well suited for the blending of elliptic quadrics primitives since they have elliptical lines of curvature. We also show how one can convert a scaled cyclide into a set of rational quadric Bézier patches.

This paper is organized as follows: section 2 presents the needed mathematical background. After this follows a brief overview of Dupin cyclides and thier underlying mathematic formulations in section 3. The bulk of the paper (sections 4 and 5) develops the mathematics and properties of scaled cyclides and the algorithm developed for their conversion into rational quadric Bézier patches. In section 6 we show and discuss some results of the use of scaled cylides as blending surfaces. The last section concludes with a discussion of this work's contribution and some suggested perspectives for future research.

2 MATHEMATICAL BACKGROUND

We note $\mathcal{E}$ the 3D Euclidian affine space with the ordinary scalar product and $\overrightarrow{\mathcal{E}}$ its associated vector space. Equations of cyclides and scaled cyclides are given into the reference $\mathcal{R}$ defined by $(O, \overrightarrow{\imath}, \overrightarrow{\jmath}, \overrightarrow{k})$. Vectors $\overrightarrow{\imath}$, $\overrightarrow{\jmath}$, and $\overrightarrow{k}$ constitute the basis of the vector space $\overrightarrow{\mathcal{E}}$. The scalar product of two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ is noted by $\langle \overrightarrow{u}, \overrightarrow{v} \rangle$.

2.1 TANGENT SPACE

A surface S in $\mathcal{E}$ may be generated by a parametric map Γ, application of a domain $\mathbb{U}$ of $\mathbb{R}^2$ into $\mathcal{E}$. Formula 1 gives the equation of such a parametric map. The surface S is considered of class $\mathcal{C}^k$ if Γ is of class $\mathcal{C}^k$, $k \in \mathbb{N}$.

$$S = \{\Gamma(u, v),\ (u, v) \in \mathbb{U}\}\,, \quad \Gamma = (\Gamma_x;\ \Gamma_y;\ \Gamma_z) \qquad (1)$$

Given $M_0 = \Gamma(u_0, v_0)$ a point on S, the tangent space to S at $\Gamma(u_0, v_0)$, denoted $\mathcal{T}_{\Gamma(u_0,v_0)}S$, is the affine space generated by $\Gamma(u_0, v_0)$ and the two vectors $\overrightarrow{t_u(u, v)} = \overrightarrow{\frac{\partial \Gamma}{\partial u}(u, v)}$ and $\overrightarrow{t_v(u, v)} = \overrightarrow{\frac{\partial \Gamma}{\partial v}(u, v)}$. Surface S is regular at point $\Gamma(u_0, v_0)$ if and only if the dimension of $\mathcal{T}_{\Gamma(u_0,v_0)}S$ is equal to 2.

Geometric properties of surface S can be obtained from the following first and second fundamental forms.

2.2 THE FIRST FUNDAMENTAL FORM

The first fundamental form $\varphi_{1,\Gamma(u_0,v_0)}$ is the restriction of the surface's tangent space at $\Gamma(u_0, v_0)$. That way the metric of the tangent space is induced by the metric of $\mathcal{E}$. We note that $\varphi_{1,\Gamma(u_0,v_0)}$ is defined to be positive. Its matrix at point $\Gamma(u_0, v_0)$ is given by equation 2 [LFA91].

$$\begin{pmatrix} E(u_0, v_0) & F(u_0, v_0) \\ F(u_0, v_0) & G(u_0, v_0) \end{pmatrix} \qquad (2)$$

where $E(u_0, v_0) = \left\| \overrightarrow{\frac{\partial \Gamma}{\partial u}(u_0, v_0)} \right\|^2$, $F(u_0, v_0) = \left\langle \overrightarrow{\frac{\partial \Gamma}{\partial u}(u_0, v_0)}, \overrightarrow{\frac{\partial \Gamma}{\partial v}(u_0, v_0)} \right\rangle$ and $G(u_0, v_0) = \left\| \overrightarrow{\frac{\partial \Gamma}{\partial v}(u_0, v_0)} \right\|^2$..

2.3 THE SECOND FUNDAMENTAL FORM

Let $\overrightarrow{N(u_0, v_0)}$ be the surface principal normal vector at point $\Gamma(u_0, v_0)$. The Weingarten endomorphism $\Psi_{\Gamma(u_0,v_0)}$ of the tangent space $\mathcal{T}_{\Gamma(u_0,v_0)}S$ is symmetric for the ordinary scalar product and may be defined as the basis vectors' images:

$$\begin{aligned} \Psi_{\Gamma(u_0,v_0)}\left(\overrightarrow{\frac{\partial \Gamma}{\partial u}(u_0, v_0)}\right) &= -\overrightarrow{\frac{\partial N}{\partial u}(u_0, v_0)} \\ \Psi_{\Gamma(u_0,v_0)}\left(\overrightarrow{\frac{\partial \Gamma}{\partial v}(u_0, v_0)}\right) &= -\overrightarrow{\frac{\partial N}{\partial v}(u_0, v_0)} \end{aligned} \qquad (3)$$

The second fundamental form $\varphi_{2,\Gamma(u_0,v_0)}$ is the form associated with the symmetric bilinear form of $\mathcal{T}_{\Gamma(u_0,v_0)}S \times \mathcal{T}_{\Gamma(u_0,v_0)}S$ in $\mathbb{R}$. It is defined by: $\varphi_{2,\Gamma(u_0,v_0)}(\overrightarrow{w}) = \langle \overrightarrow{w}, \Psi_{\Gamma(u_0,v_0)}(\overrightarrow{w}) \rangle$ for all $\overrightarrow{w} \in \mathcal{T}_{\Gamma(u_0,v_0)}S$. Then it may be expressed in terms of the first fundamental form $\varphi_{1,\Gamma(u_0,v_0)}$ as: $\varphi_{2,\Gamma(u_0,v_0)}(\overrightarrow{w}) = \varphi_{1,\Gamma(u_0,v_0)}\left(\overrightarrow{w}, \Psi_{\Gamma(u_0,v_0)}(\overrightarrow{w})\right)$.

The matrix of this second fundamental form at point $\Gamma(u_0, v_0)$ is given by formula 4 [BG92, RS97, LFA91].

$$\begin{pmatrix} L(u_0, v_0) & M(u_0, v_0) \\ M(u_0, v_0) & N(u_0, v_0) \end{pmatrix} \qquad (4)$$

where $L(u_0, v_0) = \left\langle \overrightarrow{\frac{\partial^2 \Gamma}{\partial u^2}(u_0, v_0)}, \overrightarrow{N(u_0, v_0)} \right\rangle$

$M(u_0, v_0) = \left\langle \overrightarrow{\frac{\partial^2 \Gamma}{\partial u \partial v}(u_0, v_0)}, \overrightarrow{N(u_0, v_0)} \right\rangle$, and $N(u_0, v_0) = \left\langle \overrightarrow{\frac{\partial^2 \Gamma}{\partial v^2}(u_0, v_0)}, \overrightarrow{N(u_0, v_0)} \right\rangle$.

2.4 PRINCIPAL CURVATURES AND PRINCIPAL DIRECTIONS

Under the above considerations, there is a base $\left(\overrightarrow{e_{1,\Gamma(u_0,v_0)}}, \overrightarrow{e_{2,\Gamma(u_0,v_0)}}\right)$ of the tangent space $\mathcal{T}_{\Gamma(u_0,v_0)}S$ orthonormal for $\varphi_{1,\Gamma(u_0,v_0)}$ and orthogonal for $\varphi_{2,\Gamma(u_0,v_0)}$ such that for all $\overrightarrow{w} \in \mathcal{T}_{\Gamma(u_0,v_0)}S$, we have equation 5.

$$\begin{aligned} \varphi_{2,\Gamma(u_0,v_0)}\left(\overrightarrow{e_{1,\Gamma(u_0,v_0)}}\right) &= \rho_{1,\Gamma(u_0,v_0)} \\ \varphi_{2,\Gamma(u_0,v_0)}\left(\overrightarrow{e_{2,\Gamma(u_0,v_0)}}\right) &= \rho_{2,\Gamma(u_0,v_0)} \\ \varphi_{2,\Gamma(u_0,v_0)}(\overrightarrow{w}) &= \lambda^2 \rho_{1,\Gamma(u_0,v_0)} + \mu^2 \rho_{2,\Gamma(u_0,v_0)} \end{aligned} \qquad (5)$$

where (λ, μ) are the components of $\overrightarrow{w}$ and numbers $\rho_{1,\Gamma(u_0,v_0)}$ and $\rho_{2,\Gamma(u_0,v_0)}$ are given by equation 6.

$$\rho_{i,\Gamma(u_0,v_0)} = \frac{L+N-(-1)^i\sqrt{(L-N)^2+4M^2}}{2} \quad (6)$$

Let us now introduce definitions of principal directions and principal curvatures [BG92, RS97, LFA91]:
- Principal directions of the surface S at point $\Gamma(u_0, v_0)$ are defined by vectors $\overrightarrow{e_{1,\Gamma(u_0,v_0)}}$ and $\overrightarrow{e_{2,\Gamma(u_0,v_0)}}$.
- Principal curvatures of the surface S at point $\Gamma(u_0, v_0)$ are defined by numbers $\rho_{1,\Gamma(u_0,v_0)}$ and $\rho_{2,\Gamma(u_0,v_0)}$.
- A curvature line of the surface S may be any curve on S such that the tangent to the curve at any curve point is collinear with one of the vectors defining the principal directions of S.

3 OVERVIEW OF DUPIN CYCLIDES

3.1 DEFINITIONS

Cyclide surfaces represent a family of ringed surfaces, ie. surfaces generated by a circle of variable radius sweeping through space [Pra95, CDH89]. They have circular lines of curvature, and horizontal and vertical circles of curvature are orthogonal. It is possible to formulate them either as algebraic or parametric surfaces. These are three equivalent definitions of these surfaces [CDH88]: A cyclide is the envelope of a variable sphere having its center on a given plane and touching two given sphere; A cyclide is the envelope of a variable sphere that touches three fixed spheres in a continuous manner; A cyclide is the envelope surface of a variable sphere belonging to one of the four series of spheres that touch three given spheres.

3.2 EQUATIONS

A Dupin cyclide has two equivalent implicit equations [For12]. It can be characterized either by equation (7) or (8).

$$\left(x^2+y^2+z^2-\mu^2+b^2\right)^2 = 4\left(ax-c\mu\right)^2+4b^2y^2 \quad (7)$$

$$\left(x^2+y^2+z^2-\mu^2-b^2\right)^2 = 4\left(cx-a\mu\right)^2-4b^2z^2 \quad (8)$$

Parameters a, b, and c are related by the relation $c^2 = a^2-b^2$ with $a \geq c$. Parameters a, c, and μ determine the type of the cyclide. When $c < \mu \leq a$ it is a ring cyclide. When $0 < \mu \leq c$ it is a horned cyclide. When $\mu > a$ it is a spindle cyclide.
The parametric equation of a Dupin cyclide $\Gamma(\theta,\psi) = (X(\theta,\psi), Y(\theta,\psi), Z(\theta,\psi))$ is represented by equation (9), where variables θ and ψ verify $(0 \leq \theta,\ \psi \leq 2\pi)$.

$$\Gamma(\theta,\psi) = \begin{cases} \frac{\mu(c-a\cos\theta\ \cos\psi)+b^2\cos\theta}{a-c\cos\theta\ \cos\psi} \\ \frac{b\sin\theta\ \times(a-\mu\cos\psi)}{a-c\cos\theta\ \cos\psi} \\ \frac{b\sin\psi\ \times(c\cos\theta-\mu)}{a-c\cos\theta\ \cos\psi} \end{cases} \quad (9)$$

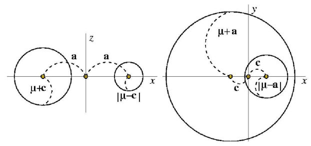


Figure 1: Sectioning a Dupin cyclide by its planes of symmetry

3.3 PROPERTIES

Presenting the whole properties of Dupin cyclides is out of the scope of this paper. Several properties of these surfaces make them particularly suitable for use in CAGD. We show in this section some important properties used in this work either for the definition of scaled cyclides, for blending or for the conversion into Bézier patches. For example:
- Simple mathematical expressions either as implicit or parametric equations.
- Circular lines of curvature. On these lines of curvature either angle θ or ψ is constant. The angle between the surface normal and the principal normal on the curvature line is also constant.
- Each cyclide have two perpendicular planes of symmetry. The intersection of a cyclide with any one of its planes of symmetry gives two circular curves called the principal circles. Figure 1 shows a cyclide section on one of its planes of symmetry.
- In the parametric form, the knowledge of principal circles allows the computation of parameters a, c, and μ. Principal circles are lines of curvature of the cyclide computed where any one of the following is true: $\psi = 0$ or $\psi = \pi$ or $\theta = 0$ or $\theta = \pi$.
- Curvature lines and tangent cones simplify the use of cyclides as blending surfaces.

4 SCALED CYCLIDES

4.1 DEFINITION

In order to generalize the use of Dupin cyclides for the blending of elliptical quadrics, we define scaled cyclides as special class of surfaces obtained from Dupin cyclides by the application of an affine scaling transformation. We choose to apply the simple transformation that converts circles into ellipses.
In the plane two such transformations are possible:
-orthogonal affinities with axis passing trough the center of the circle to be converted. - scaling applications having a center coinciding with the center of the circle to be converted. In order to obtain the simplest possible formulation of the desired surfaces, we choose to apply the scaling transformation.The application of a scaling function of parameters (X_0, Y_0, Z_0) to Dupin cyclide's implicit and parametric equations (Formulae 7, 8 and 9) allows one to define the implicit and parametric equations of scaled cyclides (Formulae

10, 11 and 12).

$$\left(\left(\frac{x}{X_0}\right)^2+\left(\frac{y}{Y_0}\right)^2+\left(\frac{z}{Z_0}\right)^2+b^2-\mu^2\right)^2 - 4\left(a\frac{x}{X_0}-c\mu\right)^2-4b^2\left(\frac{y}{Y_0}\right)^2=0 \quad (10)$$

$$\left(\left(\frac{x}{X_0}\right)^2+\left(\frac{y}{Y_0}\right)^2+\left(\frac{z}{Z_0}\right)^2-b^2-\mu^2\right)^2 - 4\left(c\frac{x}{X_0}-a\mu\right)^2+4b^2\left(\frac{z}{Z_0}\right)^2=0 \quad (11)$$

$$\Gamma(\theta,\psi)=\left(\begin{array}{l} X_0\,\dfrac{\mu(c-a\cos\theta\ \cos\psi)+b^2\cos\theta}{a-c\cos\theta\ \cos\psi}\\ Y_0\,\dfrac{b\sin\theta\ \times(a-\mu\cos\psi)}{a-c\cos\theta\ \cos\psi}\\ Z_0\,\dfrac{b\sin\psi\ \times(c\cos\theta-\mu)}{a-c\cos\theta\ \cos\psi}\end{array}\right) \quad (12)$$

where X_0, Y_0 and Z_0 are positive real numbers. Figure 2 shows an example of a Dupin cyclide and its equivalent scaled cyclide. Coefficients of the scaling transformation are: $X_0 = 3, Y_0 = 3$ and $Z_0 = 2$.

Figure 2: Left: A Dupin cyclide Surface. Right: The equivalent scaled cyclide

4.2 SCALING AND PARTIAL DERIVATIVES

In this section, we first analyze effects of scaling transformations onto the partial derivatives and gradient in the case of a scaled general surface, in order to deduce some computation expressions. Second, we show how partial derivatives and gradient can be computed in the specific case of a scaled cyclide.Let us consider a surface S of $\mathcal{E}$ with its parametric map Γ defined onto a parametric domain $\mathbb{U}$ of $\mathbb{R}^2$ and an affine application f used to define a surface $S' = f(S)$. The parametric map Γ' of S' is then defined by $\Gamma' = f \circ \Gamma$. We call $\overrightarrow{f}$ the linear application associated to f.Let $X_0,\ Y_0$ and Z_0 be three positive floats, $\mathcal{M}_f$ the matrix of the affine scaling application f and $\mathcal{M}_{\overrightarrow{f}}$ the matrix of the linear application associated to f, (formula 13).

$$\mathcal{M}_f=\mathcal{M}_{\overrightarrow{f}}=\begin{pmatrix}X_0&0&0\\0&Y_0&0\\0&0&Z_0\end{pmatrix} \quad (13)$$

The rank of the application $\overrightarrow{f}$ is 3, then $f(\mathcal{E}) = \mathcal{E}$ and the image of a plane by f is another plane.

Partial derivatives of Γ' can be computed directly from derivatives of Γ and matrix $\mathcal{M}_{\overrightarrow{f}}$ using equation 14.

$$\begin{array}{l}\overrightarrow{\dfrac{\partial\Gamma'}{\partial u}(u_0,v_0)}=\mathcal{M}_{\overrightarrow{f}}\overrightarrow{\dfrac{\partial\Gamma}{\partial u}(u_0,v_0)}\\ \overrightarrow{\dfrac{\partial\Gamma'}{\partial v}(u_0,v_0)}=\mathcal{M}_{\overrightarrow{f}}\overrightarrow{\dfrac{\partial\Gamma}{\partial v}(u_0,v_0)}\end{array} \quad (14)$$

In the same way, one can compute the gradient of Γ' using equation 15, where W is a diagonal matrix given by:

$$W=\begin{pmatrix}Y_0Z_0&0&0\\0&X_0Z_0&0\\0&0&X_0Y_0\end{pmatrix}.$$

$$\overrightarrow{\nabla\Gamma'(u_0,v_0)}=W\overrightarrow{\nabla\Gamma(u_0,v_0)} \quad (15)$$

Second partial derivatives of Γ' can also be computed directly from second derivatives of Γ.

$$\begin{array}{l}\overrightarrow{\frac{\partial^2\Gamma'}{\partial u^2}(u_0,v_0)}=\mathcal{M}_{\overrightarrow{f}}\overrightarrow{\frac{\partial^2\Gamma}{\partial u^2}(u_0,v_0)}\\ \overrightarrow{\frac{\partial^2\Gamma'}{\partial u\partial v}(u_0,v_0)}=\mathcal{M}_{\overrightarrow{f}}\overrightarrow{\frac{\partial^2\Gamma}{\partial u\partial v}(u_0,v_0)}\\ \overrightarrow{\frac{\partial^2\Gamma'}{\partial v^2}(u_0,v_0)}=\mathcal{M}_{\overrightarrow{f}}\overrightarrow{\frac{\partial^2\Gamma}{\partial v^2}(u_0,v_0)}\end{array} \quad (16)$$

Partial derivatives of a scaled cyclide Γ at a point $\Gamma(\theta_0,\psi_0)$ can be computed using vectors from the base of the tangent space $\mathcal{T}_{\Gamma(\theta_0,\psi_0)}S$. In the case of a scaling transformation, this base is composed of two generator vectors $\overrightarrow{t_1(\theta_0,\psi_0)}$ and $\overrightarrow{t_2(\theta_0,\psi_0)}$, equation 17, which one can define directly from equation 14.

$$\begin{array}{l}\overrightarrow{t_1(\theta_0,\psi_0)}=\begin{pmatrix}b^2X_0\sin\theta_0\\ Y_0b\,(c\cos\psi_0-a\cos\theta_0)\\ Z_0bc\sin\theta_0\ \sin\psi_0\end{pmatrix}\\ \overrightarrow{t_2(\theta_0,\psi_0)}=\begin{pmatrix}b^2X_0\cos\theta_0\ \sin\psi_0\\ Y_0ab\sin\theta_0\sin\psi_0\\ Z_0b\,(c\cos\theta_0-a\cos\psi_0)\end{pmatrix}\end{array} \quad (17)$$

Equation 18 gives first order partial derivatives of the scaled cyclide. Equations of the gradient and second order partial derivatives can be obtained in the same way from equations 15 and 16.

$$\begin{array}{l}\overrightarrow{\dfrac{\partial\Gamma}{\partial\theta}(\theta_0,\psi_0)}=\dfrac{\mu\cos\psi_0-a}{(a-c\cos\theta_0\ \cos\psi_0)^2}\overrightarrow{t_1(\theta_0,\psi_0)}\\ \overrightarrow{\dfrac{\partial\Gamma}{\partial\psi}(\theta_0,\psi_0)}=\dfrac{\mu-c\cos\theta_0}{(a-c\cos\theta_0\ \cos\psi_0)^2}\overrightarrow{t_2(\theta_0,\psi_0)}\end{array} \quad (18)$$

4.3 SCALING AND LINES OF CURVATURE

The purpose of this section is to proove that principal directions are invariant by a scaling transformation, and thus

curvature lines of the scaled surface are the images of curvature lines of the original surface.
As shown by equation 19, coefficients of the second fundamental form of the scaled cyclide may be computed by multiplying coefficients of the general second fundamental form by $\frac{X_0 Y_0 Z_0}{\left\|\overrightarrow{\nabla\Gamma'(u_0,v_0)}\right\|}$.

$$\begin{aligned} \varphi'_{2,\Gamma'(u_0,v_0)} &= \frac{X_0 Y_0 Z_0}{\left\|\overrightarrow{\nabla\Gamma'(u_0,v_0)}\right\|}\varphi_{2,\Gamma(u_0,v_0)} \\ \rho'_{i,\Gamma'(u_0,v_0)} &= \frac{X_0 Y_0 Z_0}{\left\|\overrightarrow{\nabla\Gamma'(u_0,v_0)}\right\|}\rho_{i,\Gamma(u_0,v_0)} \end{aligned} \tag{19}$$

Given two normalized vectors $\overrightarrow{v_{1,\Gamma(u_0,v_0)}}$ and $\overrightarrow{v_{2,\Gamma(u_0,v_0)}}$ defining principal directions of a surface S and belonging to the tangent space $\mathcal{T}_{\Gamma(u_0,v_0)}S$, there are real numbers x_i and y_i with $i \in \{1,2\}$ such that $\overrightarrow{v_{i,\Gamma(u_0,v_0)}} = x_i\overrightarrow{\frac{\partial\Gamma}{\partial u}(u_0,v_0)} + y_i\overrightarrow{\frac{\partial\Gamma}{\partial v}(u_0,v_0)}$. In order that $\overrightarrow{v_{i,\Gamma(u_0,v_0)}}$ be one of the two vectors that give the i^{th} principal direction, it suffices to have:

$$\begin{aligned} &(L(u_0,v_0) - \rho_1(u_0,v_0))\,x_1 + M(u_0,v_0)\,y_1 = 0 \\ &M(u_0,v_0)\,x_2 + (L(u_0,v_0) - \rho_2(u_0,v_0))\,y_2 = 0 \end{aligned} \tag{20}$$

If $\Gamma(u_0,v_0)$ is an umbilicus point, formula 6 implies that $M(u_0,v_0) = 0$ and $\rho_{i,\Gamma(u_0,v_0)} = L(u_0,v_0) = N(u_0,v_0)$. Then equation 20 shows that we cannot determine the principal directions. We use a parametrisation of the cyclide such as the lines of curvature are obtained with one of the parameters constant. The ombilicus points of a Dupin cyclide are isolated. The problem is then solved.
Let $\overrightarrow{w} \in \mathcal{T}_{\Gamma(u_0,v_0)}S$ such that $\overrightarrow{w} = x\overrightarrow{\frac{\partial\Gamma}{\partial u}(u_0,v_0)} + y\overrightarrow{\frac{\partial\Gamma}{\partial v}(u_0,v_0)}$. As $\overrightarrow{f}$ is a linear application, we have $\overrightarrow{w'} = \overrightarrow{f}(\overrightarrow{w}) \in \mathcal{T}_{\Gamma'(u_0,v_0)}S'$ and $\overrightarrow{w'} = x\overrightarrow{f}\left(\overrightarrow{\frac{\partial\Gamma}{\partial u}(u_0,v_0)}\right) + y\overrightarrow{f}\left(\overrightarrow{\frac{\partial\Gamma}{\partial v}(u_0,v_0)}\right)$. According to formula 14 we get $\overrightarrow{w'} = x\overrightarrow{\frac{\partial\Gamma'}{\partial u}(u_0,v_0)} + y\overrightarrow{\frac{\partial\Gamma'}{\partial v}(u_0,v_0)}$. In order that vector $\overrightarrow{w}$ be one of the vectors that generate a principal direction of S', coefficients x and y must verify the formula 20. According to the relation between the second fundamental forms (formula 19), to have vector $\overrightarrow{v_{i,\Gamma(u_0,v_0)}} = x'_i\overrightarrow{\frac{\partial\Gamma'}{\partial u}(u_0,v_0)} + y'_i\overrightarrow{\frac{\partial\Gamma'}{\partial v}(u_0,v_0)}$ as one of the two vectors giving the i^{th} principal direction of S' at $\Gamma'(u_0,v_0)$, it suffices that x'_1 and y'_1 verify equation 21.

$$\begin{aligned} &(L(u_0,v_0) - \rho_1(u_0,v_0))\,x'_1 + M(u_0,v_0)\,y'_1 = 0 \\ &M(u_0,v_0)\,x'_1 + (L(u_0,v_0) - \rho_2(u_0,v_0))\,y'_2 = 0 \end{aligned} \tag{21}$$

This means that principal directions are invariant by a scaling transformation.
Curvature lines of Dupin cyclides are isoparametric circles that one can obtain by keeping one of the parameters θ or ψ constant while varying the other. The image of a circle, by the scaling application described above, is an ellipse, and thus curvature lines of a scaled Dupin cyclide are ellipses. Figure 3 shows some curvature lines computed on a scaled cyclide. For the left sub-figure the variable θ is constant, while for the right sub-figure the constant variable is ψ

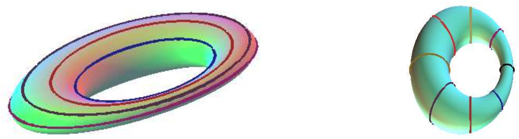


Figure 3: Curvature lines of scaled cyclides

4.4 SCALED CYCLIDES PLANES OF SYMMETRY

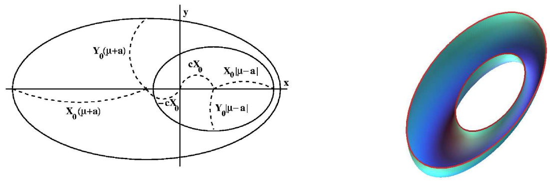


Figure 4: Intersection of a scaled cyclide and plane :$z = 0$

Like Dupin cyclides, the proposed scaled cyclides have planes $y = 0$ and $z = 0$ as planes of symmetry. The intersection of scaled cyclides with any one of these planes corresponds to constant values of parameters ψ and θ and gives two ellipses. Figure 4 illustrates the section of a scaled cyclide with the planes $z = 0$. The intersection is composed of the union of two ellipses defined by centers $\Omega_z^1(cX_0,0,0)$ and $\Omega_z^2(-cX_0,0,0)$ and axis $X_0|a-\mu|$, $Y_0|a-\mu|$, and $X_0(a+\mu)$, $Y_0(a+\mu)$. By a similar reasoning one can obtain the equivalent construction for the intersection of the scaled cyclide and the plane $y = 0$.

5 CONVERTING SCALED CYCLIDES INTO BÉZIER PATCHES

Rational Quadric Bézier (RQB) surfaces are parametric surfaces of degree 2 widely used in geometric modeling and shape design. Several books and papers have studied these surfaces and analyzed their properties from mathematical to CAGD points of view [FP79, Far93, Hof89, B86, AR90].
A point $M(u,v)$ on a RQB surface verifies the following expression:

$$\overrightarrow{OM(u,v)} = \frac{\sum_{i=0}^{2}\sum_{j=0}^{2} w_{ij}B_i(u)B_j(v)\overrightarrow{OP_{ij}}}{\sum_{i=0}^{2}\sum_{j=0}^{2} w_{ij}B_i(u)B_j(v)} \tag{22}$$

where $(u, v) \in [0;1]^2$ are surface parameters, O is a point of the 3D Euclidian space $\mathcal{E}$, P_{ij} are the set of 3D points that define the control polygon of the surface, w_{ij} are weights associated to control points P_{ij}, and $B_i(t)$ are second degree Bernstein blending polynomials defined by:

$$B_0(t) = (1-t)^2 \quad B_1(t) = 2t\,(1-t) \quad B_2(t) = t^2 \qquad (23)$$

Isoparametric curves that one can trace on a RQB surface while keeping one variable constant and varying the other one are conics. Dupin cyclide curvature lines are also conics (circles), so it is possible to convert a Dupin cyclide into a set of RQB surfaces [Pra90]. On the other hand, RQB curves and surfaces are invariant under affine or projective transformations [DP98], so we can convert a scaled cyclide, or a portion of a scaled cyclide, into RQB surfaces in the same way as Dupin cyclides. Suppose that the values of parameters θ and ψ that delimit the portion of the Dupin cyclide to be converted are θ_0, θ_1, ψ_0, and ψ_1. Then we have only to define the nine control points $(P_{ij})_{0\leq i,j\leq 2}$ and their associated weights $(w_{ij})_{0\leq i,j\leq 2}$ [Pra90]. Our weigths are the absolu values of Pratt's weights. The control points of a scaled cyclide are $f\,(P_{ij})_{0\leq i,j\leq 2}$ where $f(x,y,z) = (X_0 x, Y_0 y, Z_0 z)$. Figure 5 shows a scaled cyclide converted into 9 RQB patches.

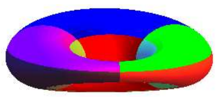


Figure 5: A scaled cyclide converted into 9 RQB surfaces

6 QUADRICS BLENDING USING SCALED CYCLIDES

In this section, we will show how scaled cyclides can be used for the blending of elliptic quadric primitives. First we will give the general steps one can follow to construct the blending scaled cyclide (or the portion of a scaled cyclide). Second, we will show the complete processes for the blending of two cylinders and for the blending of a cylinder and a plane. These two specific blending cases can be easily generalized to blend other elliptic quadrics. Some blending examples will be given and commented.

The blending algorithm is composed of 4 steps as follow:
- Expressing equations of primitives to be joined into a unique reference. - Computing ellipses tangent to the primitives. These ellipses will help to construct the blending scaled cyclide. - Finding parameters of the cyclide. - Constructing the blending scaled cyclide and trimming it to keep only the useful part

6.1 BLENDING TWO CYLINDERS

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two cylinders both lying in the same half space delimited by a plane $\mathcal{P}$. We will construct the blending scaled cyclide such that the plane $\mathcal{P}$ will be its plane of symmetry. We choose $\mathcal{P}$ to be the plane $(y = 0)$. This helps to reduce computations. A similar reasoning may be applied for the plane $z = 0$, but more analysises must be done for the blending of two cylinders in a general position. In the plane $y = 0$, the equations of cylinders $\mathcal{C}_1$ and $\mathcal{C}_2$ are given by formulae 24 and 25 respectively where θ and v verify $\theta \in [0; 2\pi]$ $v \geq 0$, and a_1, b_1, a_2 and b_2 are positive real numbers.

$$\mathcal{C}_1(\theta, v) = \begin{cases} -x_1 + a_1 \cos(\theta) \\ v \\ b_1 \sin(\theta) \end{cases} \qquad (24)$$

$$\mathcal{C}_2(\theta, v) = \begin{cases} x_1 + a_2 \cos(\theta) \\ v \\ b_2 \sin(\theta) \end{cases} \qquad (25)$$

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two ellipses defined by $\mathcal{E}_i = \mathcal{C}_i \cap \mathcal{P},\ i \in \{1;2\}$, such that one axis of $\mathcal{E}_1$ is coincident with one axis of $\mathcal{E}_2$ (Figure 6). These ellipses are defined to be the principal ellipses of the scaled cyclide we are constructing.

Under these considerations, one can use equations of el-

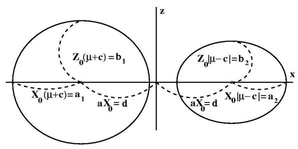


Figure 6: Construction used for the blending of two cylinders

lipses $\mathcal{E}_1$ and $\mathcal{E}_2$ and equations of scaled cyclides to compose the following system of equations :

$$\begin{cases} 2aX_0 = d \\ X_0(c+\mu) = a_1 \\ Z_0(c+\mu) = b_1 \\ X_0(\mu - c) = a_2 \\ Z_0(\mu - c) = b_2 \end{cases} \qquad (26)$$

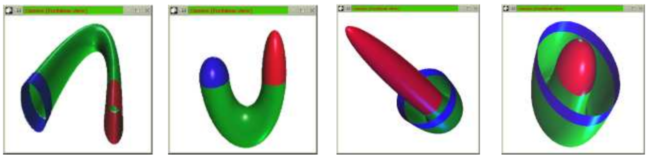
where d is referred to as the distance between centers of $\mathcal{E}_1$ and $\mathcal{E}_2$. This system solves for $X_{0,t}, Z_{0,t}, a_t, \mu_t$, and c_t. The obtained solution is presented in 27 with $t \in \mathbb{R}_*^+$.

$$(X_{0,t}, Z_{0,t}, a_t, \mu_t, c_t) = \left(t, \frac{b_1 t}{a_1}, \frac{d}{2t}, \frac{a_1 + a_2}{2t}, \frac{a_1 - a_2}{2t}\right) \qquad (27)$$

This solution allows one to define a family of ring or spindle scaled cyclides. The type of this cyclide is a function of the position of cylinders $\mathcal{C}_1$ and $\mathcal{C}_2$:
- If $a_1 + a_2 \leq d$ then we obtain a family of ring scaled cyclides.
- If $a_1 + a_2 > d$ and $d \geq a_1 - a_2$ we obtain a family of spindle scaled cyclides.

Left Pictures of igure 7 shows some blending results. The left part represents the blending of two cylinders. The right part concerns the blending of a cylinder and an ellipsoid.

A similar analysis as above may be done for the blending of two primitives positioned in the same half space of plane $z = 0$. Right pictures of Figure 7 gives two different blendings of a cylinder and an ellipsoid.



Plane $y = 0$ Plane $z = 0$

Figure 7: Blending by the plane $y = 0$ or $z = 0$

6.2 BLENDING A CYLINDER AND A PLANE

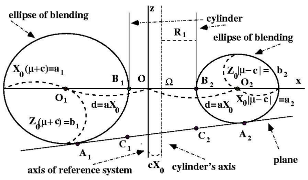


Figure 8: Construction of the tangent ellipses

Let P be a plane of equation $z = mx + p$ and an elliptic cylinder $\mathcal{C}$ defined by the elliptic base Ω, the half-axes R_1 and R_2, and the principal axis Δ. We need to express equations of the plane and the cylinder in the reference $\left(\Omega, \overrightarrow{\imath}, \overrightarrow{\jmath}, \overrightarrow{k}\right)$, where $\overrightarrow{\imath} = \frac{1}{\Omega B_2}\overrightarrow{\Omega B_2}$, $\overrightarrow{\jmath} = \frac{1}{\Omega B'}\overrightarrow{\Omega B'}$, and $\overrightarrow{k}$ is the direction of Δ. In order to construct the blending cyclide we must determine two ellipses γ_1 and γ_2 called blending ellipses (Figure 8). These are some constraints for correctly defining γ_1 and γ_2:

- Their centers are $O_1\,(x_1,\,0,\,0)$ and $O_2\,(x_2,\,0,\,0)$ respectively.

They pass through points B_1, A_1 and B_2, A_2 respectively.

- They are tangent to the plane on A_1 and A_2 respectively.
- They are tangent to the cylinder on B_1 and B_2 respectively.

We define points $C_1\,(-R_1,\,0,\,-mR_1 + p)$ and $C_2\,(R_1,\,0,\,mR_1 + p)$ such that for $i \in \{1;2\}$, C_i is the intersection between the cylinder edge passing through point B_i and the plane. Ellipses γ_i will then be modelled by a rational quadric Bézier defined by control points A_i, B_i, and C_i.

Under these considerations coefficients of the scaled cyclide that blend the two primitives are given by:

$$a = d, \;\; \mu = d - R_1, \;\; c = a_1 + R_1 - d,$$

$$Y_0 = \frac{R_2}{R_1}, \;\; Z_0 = \frac{b_1}{a_1} = \frac{b_2}{a_2}, \;\; b = \sqrt{a^2 - c^2}$$

placing these coefficients into the parametric equation of scaled cyclide (equation 12) gives the blending scaled cyclide.

Left pictures of Figure 9 shows results of the blending of a plane and a cylinder and the blending of a plane and an ellipsoid. Right pictures of Figure 9 shows a more complicated example where two scaled cyclides are combined to construct the blending of a cylinder and a plane. In figure 10, we present two views of a blending scaled cyclide converted into three Bézier surfaces. Other examples showing the blending of a cylinder and a cone, and the blending of two cones are given in Figure 11.

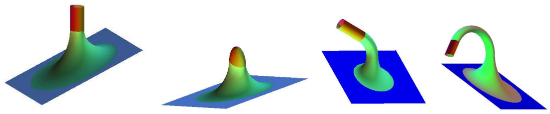


Figure 9: Plane-cylinder and plane-ellipsoid blending

Figure 10: Conversion of a blending scaled cyclide into three RQB surfaces

Figure 11: Left. cylinder-cone blending. Right. cone-cone blending

7 CONCLUSION

In this paper we have proposed a new class of low degree algebraic surfaces issued from Dupin cyclides. The proposed surfaces are images of Dupin cyclides under a scaling transformation. The well known properties of Dupin cyclides are conserved in these new surfaces. The scaling transformation ensures the obtaining of elliptic lines of curvatures that allow the use of scaled cyclides as tools for the blending of non circular quadrics primitives.

The well established conversion of scaled cyclides into rational quadric Bézier surfaces bridges the gap between these two modeling tools and may accelerate the introduction of scaled cyclides into geometric modeling systems based on

classical parametric surfaces.
Further investigations need to be done in order to generalize our blending construction to the cases of other primitives.

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