ZERO VARIANCE IMPORTANCE SAMPLING DRIVEN POTENTIAL TRACING ALGORITHM FOR GLOBAL ILLUMINATION

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ABSTRACT

This paper presents a novel Monte Carlo strategy for solving the global illumination problem. The unusual Monte Carlo approach is different from the one, which is utilized widely in prevalent global illumination algorithms at present, and breaks a new Monte Carlo path for settling the problem. In this way, plenty of unbiased estimators can be employed to enrich the solutions so as to lead to simple error control, and also various variance reduction techniques can be applied conveniently to speed up the estimation at little additional cost. Especially, an implementation of a theoretically zero variance importance sampling driven potential tracing algorithm on premise of the new scheme has been proposed and carried out. Results having been obtained and comparisons with traditional algorithms show that this new framework and new algorithm is very promising.

Keywords: global illumination, Monte Carlo, zero variance importance sampling, random walk.

1. INTRODUCTION

As we have known, the rendering equation [Kajiy86]

$$L(x,\Theta_x) = L_e(x,\Theta_x) + \int_{1/\Omega_x} L(y,\Theta_y) f_r(\Theta_y, x, \Theta_x) \left|\cos(N_x,\Theta_y)\right| d\omega_y \quad (1)$$

and the potential equation [Patta93a]

$$P(x,\Theta_x) = P_e(x,\Theta_x) + \int_{\Omega_y} P(y,\Theta_y) f_r(\Theta_x, y, \Theta_y) \left|\cos(N_y,\Theta_y)\right| d\omega_y \quad (2)$$

depict the light transport from two different points of view. The calculation of flux from a small region in a small spread of directions is the main purpose of illumination computations. Expression of the flux

$$\Phi(S) = \int_A \int_{\Omega_x} L(x,\Theta_x) P_e(x,\Theta_x) \left|\cos(N_x,\Theta_x)\right| d\omega_x dx \quad (3)$$

could have one form of integral Eq. 3 dependent on the rendering equation and another form of integral Eq. 4 dependent on the potential equation [Patta95]. The flux to be solved along with its corresponding equation defines the global illumination. It should be pointed out that the global illumination problem related to the rendering equation is adjoint to that related to the potential equation [Kadib96].

$$\Phi(S) = \int_A \int_{\Omega_x} L_e(x,\Theta_x) P(x,\Theta_x) \left|\cos(N_x,\Theta_x)\right| d\omega_x dx \quad (4)$$

In theory, the rendering equation and the potential equation belong to the Fredholm integral equation of the second kind; also, global illumination can be regarded as the transport problem [Arvoj93]. According to the computational approaches, finite element and Monte Carlo are the principal methods to solve the global illumination. Due to generality and simplicity of Monte Carlo, research on global illumination has been emphasized on the stochastic methods or the random walk methods [Veach97a].

In this paper we firstly introduce an alternative computational framework, which is used for solving the general transport problem and different from the one that is widely utilized in the existing Monte Carlo global illumination algorithms, into the field of global illumination calculation. On the basis of this new computing method, unbiased estimators are available in abundance to make better solutions to global illumination and to achieve error reduction easily. Secondly, we propose a new general Monte Carlo global illumination algorithm from taking full advantage of zero variance importance sampling procedure under the new computational framework. The new algorithm pays much attention to solving the global illumination as a whole rather than only seeks after optimized soluitions to the rendering equation or to the potential equation. In this way sampling is more efficient than before without any bias.

2. PREVIOUS MONTE CARLO METHODS FOR GLOBAL ILLUMINATION AND THEIR DEFECTS

Much has been done on Monte Carlo or random walk to simulate global illumination in about 20 overpast years. Gathering type and shooting type algorithms [Patta93b] are the two kinds of random walk method for the global illumination depending on solving the rendering equation and the potential equation respectively. The most important thing for random walk algorithm is to optimally choose the transition probability density function (PDF) to sample the state of random walk path. Distributed ray tracing [Cookr84], path tracing [Kajiy86], particle tracing [Patta92], light tracing [Dutre93] and bi-directional path tracing [Lafor93, Veach94], are all based on BRDF sampling. Shirley [Shirl96] and Dutre [Dutre94] improved sampling depending upon light sources. Jensen used the product of BRDF and approximated incoming radiance as PDF to find the random walk path [Jense95] and similar idea was also be adopted in Laforture's "5D tree" [Lafor95], in Pattanaik's importance driven particle tracing [Patta93b] and in Szirmay-kalos' importance driven quasi-Monte Carlo random walk [Szirm99]. Multiple sampling algorithm [Veach95] and Metropolis light transport algorithm [Veach97b] introduce adaptive stratagies to reduce the statistical error.

If we consider the global illumination in the form of the abstract transport problem

$$\Phi(x) = \int K(x, x_1)\Phi(x_1)dx_1 + \sigma(x) \quad (5)$$

$$I = \int \Phi(x)g(x)dx \quad (6)$$

all the algorithms mentioned above settle the problem by way of sampling point x_i from the known function $g(x)$ in Eq. 6 and evaluating the unknown $\Phi(x_i)$ using its estimator $<\Phi(x_i)>$, which is obtained through random walk simulation for Eq.5, to estimate the required integral I in Eq. 6. This way is popularized [Rubin81], moreover, it is the only one employed widely in general Monte Carlo global illumination algorithms currently [Patta95]. However, the above method can bring about several inherent deficiencies difficult to be overcome. On the one hand the current alogorithms only think of the known function $g(x)$ rather than the product of $g(x)$ and $\Phi(x_i)$ as the PDF to sample the random point x_i, which is used to estimate $<\Phi(x_i)>$ and the integral I to be solved. The one and only thing that the current algorithms are interested in is how to estimate $<\Phi(x_i)>$ better and in this sense we say that the global illumination is not solved as a whole. Of course this is far from the requirement of zero variance importance sampling to estimate the required integral I, which is the final thing we want to know in the filed of global illumination. On the other hand, even though we can draw point x_i in accordance with importance sampling, valid variance is impossible because the required integral I has to be evaluated from the uncertain random variable $<\Phi(x_i)>$. Actually, none of the zero variance estimators, which bases upon the computational approach of solving the transport problem prevalent in nowaday Monte Carlo global illumination algorithms, has been found theoretically up to now.

3. A NEW COMPUTING APPROACH FOR MONTE CARLO GLOBAL ILLUMINATION

To avoid non zero variance estimation of the global illumination problem, some way has to be developed to remodel the present computational framework for the stochastic global illumination. Fortunately, the global illumination problem can be solved via powerful Monte Carlo techniques served for nuclear engineering in a long history and this is the beginning of the so-called new approach proposed by us.

Another stochastic computational method for the general transport problem, which is distinct from the one that is used in current random walk global illumination algorithms, can be exploited to meet zero variance importance sampling and we call this the new approach. The new approach [Spani98] regards $\Phi(x)$ as the PDF and $g(x)$ as the evaluating function to estimate the required integral I in Eq. 6 of the transport problem. The procedure of sampling $\Phi(x)$ is in reality the random walk simulation for the integral equation of the second kind, namely, Eq. 5. The initial state x_0 is sampled from the source function $\sigma(x_0)$, the intermediate state x_i ($1 \le i \le L$) is sampled from the probability density

$$\frac{K(x_i, x_{i-1})}{\int K(x_i, x_{i-1})dx_i}$$

and the terminal probability of state x_L is $1 - \int K(x, x_L)dx$.

Two most common estimators for I are the collision estimator $< I >_C$ and the terminal estimator $<I>_T$: $\langle I \rangle_C = \sum_{i=1}^{L} g(x_i)$, $\langle I \rangle_T = \frac{g(x_L)}{q(x_L)}$

where $q(x_L)$ is the probability of termination at state

x_L.

Under the new approach, a conclusion for zero variance importance sampling estimator is as follows [Covey67, Laiys98]:

$\Phi^*(x)$, which is adjoint to $\Phi(x)$ and satisfies the dual integral equation

$$\Phi^*(x) = \int K^*(x, x_1)\Phi^*(x_1)dx_1 + g(x),$$

is the importance function of the entire computing procedure for the transport problem Eq. 5 and Eq. 6; the choices

$$p(x_0) = \frac{\Phi^*(x_0)\sigma(x_0)}{\int \Phi^*(x_0)\sigma(x_0)dx_0};$$

$$p(x_i, x_{i-1}) = \frac{K^*(x_{i-1}, x_i)\Phi^*(x_i)}{\Phi^*(x_{i-1})};$$

$$q_1(x_L) = \frac{g(x_L)}{\Phi^*(x_L)}$$

for generating, respectively, the initial state, transition from x_{i-1} to x_i, and termination at x_L, produce Var[$\langle I\rangle^*_T$] = 0 and Var[$\langle I\rangle^*_C$] <= I^2, where $\langle I\rangle^*_T$ and $\langle I\rangle^*_C$ are

$$\langle I\rangle^*_T =$$

$$\frac{g(x_L)}{q_1(x_L)} \frac{K^*(x_{L-1}, x_L)\cdots K^*(x_0, x_1)}{p(x_L, x_{L-1})\cdots p(x_1, x_0)} \frac{\sigma(x_0)}{p(x_0)},$$

$$\langle I\rangle^*_C =$$

$$\sum_{K=0}^{L} g(x_K) \frac{K^*(x_{K-1}, x_K)\cdots K^*(x_0, x_1)}{p(x_K, x_{K-1})\cdots p(x_1, x_0)} \frac{\sigma(x_0)}{p(x_0)}.$$

From these, one has broad and nice choices to settle the global illumination problem not only because so many kinds of estimator have been found but also the zero variance importance sampling procedure under the new approach exists.

4. ZERO VARIANCE IMPORTANCE DRIVEN POTENTIAL TRACING (ZVIDPT) ALGORITHM

To avoid tessellation of the virtual scene, we apply the new approach to developing view dependent algorithm now and the new approach can be expediently popularized to view independent scheme with some adaptive strategy. Since the global illumination can be described in the form relating to the rendering equation or the potential equation, the new approach we have introduced leads directly to two Monte Carlo global illumination algorithms. The one called potential tracing is to make use of the potential equation to construct random walk paths; the other one whose sampling procedure depends on the rendering equation is named for radiance tracing. As a generalized computing framework, the new approach makes both the potential tracing and the radiance tracing highly general.

Relying on the importance function (i.e. the radiance function L in Eq. 1), the procedure of creating random walk paths using the potential equation for ZVIDPT is as follows:

initial state is selected using

$$p(x, \Theta_x) = \frac{P_e(x, \Theta_x)L(x, \Theta_x)\left|\cos(N_x, \Theta_x)\right|}{\iint P_e(x, \Theta_x)L(x, \Theta_x)\left|\cos(N_x, \Theta_x)\right| dx d\omega_x};$$

middle state is sampled from

$$p[(x_i, \Theta_{x_i}), (x_{i-1}, \Theta_{x_{i-1}})] = \frac{f_r(\Theta_{x_i}, x_{i-1}, \Theta_{x_{i-1}})\left|\cos(N_{x_{i-1}}, \Theta_{x_i})\right| L(x_i, \Theta_{x_i})}{L(x_{i-1}, \Theta_{x_{i-1}})\left|\cos(N_{x_{i-1}}, \Theta_{x_{i-1}})\right|} \quad (7)$$

Actually, the sampled direction is chosen using this PDF Eq. 7 and x_i is the first intersected point between the ray starting out from point x_{i-1} along the opposite sampled direction and the surrounding surfaces; the probability of termination is $q(x, \Theta_x) = L_e(x, \Theta_x)/L(x, \Theta_x)$. The terminal estimator and the collision estimator of the required flux for ZVIDPT are

$$ZVIDPT_T =$$

$$\frac{L_e(x_l, \Theta_{x_L})}{q(x_L, \Theta_{x_L})} \times \frac{P_e(x_0, \Theta_{x_0})}{p(x_0, \Theta_{x_0})} \times$$

$$\frac{K^*(x_{L-1}, x_L)}{p[(x_L, \Theta_{x_L}), (x_{L-1}, \Theta_{x_{L-1}})]} \times \cdots$$

$$\times \frac{K^*(x_0, x_1)}{p[(x_1, \Theta_{x_1}), (x_0, \Theta_{x_0})]}$$

$$ZVIDPT_C =$$

$$\sum_{K=0}^{L} \frac{K^*(x_{K-1}, x_K)}{p[(x_K, \Theta_{x_K}), (x_{K-1}, \Theta_{x_{K-1}})]} \times \cdots$$

$$\times \frac{K^*(x_0, x_1)}{p[(x_1, \Theta_{x_1}), (x_0, \Theta_{x_0})]} \times$$

$$\frac{P_e(x_0, \Theta_{x_0})}{p(x_0, \Theta_{x_0})} L_e(x_K, \Theta_{x_K}),$$

where

$$K^*(x_{i-1}, x_i) = \frac{f_r(\Theta_{x_i}, x_{i-1}, \Theta_{x_{i-1}})\left|\cos(N_{x_{i-1}}, \Theta_{x_i})\right|}{\left|\cos(N_{x_i}, \Theta_{x_i})\right|}.$$

Because the light source is relatively small compared with the virtual scene, appearance of the emitted radiance Le in the two estimators and terminal probability will slow the rate of convergence. The direct illumination function $L_1(x,\Theta_x)=\iint L_e(y,\Theta_y)\left|\cos(N_x,\Theta_y)\right|dyd\omega_y$ is used in the two estimators to speed up calculation and this is exactly the application of next event estimation [Covey67]. Additionally, we creatively introduce the direct illumination into computing the terminal probability and experiments have indicated its efficiency and rationality.

In order to use the unknown radiance function as importance for ZVIDPT to optimize the procedure of sampling states for random walk, we turn to a preprocessed pass to get approximate radiance function. Owing to the capability in estimating the radiance of any point in the scene from the data structure entitled photon map, which is built up by means of particle tracing, we treat the particle tracing and constrcution of photon map as the first pass in ZVIDPT.

In sum, our ZVIDPT can be splitted into two main steps:

- Construction of the photon map.
- Sampling random walk paths and rendering image in accordance with the PDF and estimator mentioned above.

4.1 Sampling of the Initial State

In fact, the starting point is drawn upon

$$p(x,\Theta_x)=\frac{P_e(x,\Theta_x)L_{appr}(x,\Theta_x)\left|\cos(N_x,\Theta_x)\right|}{\iint P_e(x,\Theta_x)L_{appr}(x,\Theta_x)\left|\cos(N_x,\Theta_x)\right|dxd\omega_x} \quad (8)$$

where L_{appr} is the approximate radiance function. Here P_e is about the pixel as a result of view dependent ZVIDPT, and the essence of sampling initial state is the pixel sampling. We construct the piecewise constant approximation for the PDF Eq. 8 on a two dimensional n ×m grid to sample the pixel. The pixel to be sampled is divided into n ×m subpixels and each element of the grid is of onetoone correspondence with each subpixel. The ray starting from viewpoint to the uniformly sampled point of subpixel could have intersected points with the scene surfaces. Radiance of the first intersected point can be got from information contained in the photon map. The value of this radiance multiplying with the potential of the first intersected point and the cosine of angle is recorded in the grid element correspondent to the subpixel. If the record for a grid element is too little, a small fraction of the overall values stored within the whole grid is added to avoid bias. The finished grid stands for the discrete distribution of PDF and is enough for pixel sampling. When a grid element is sampled using a random number, the subpixel correspondent to this element is sampled exactly based on the PDF that is just needed in Eq. 8. The pixel sampling is final done after uniformly sampling the subpixel we have chosen.

4.2 Sampling of the Intermediate State

The PDF in Eq. 7 for sampling the intermediate state of random walk is a product of several items. Some standard sampling techniques for multiplicator of distributions [Kalos86] can be used here; also, the approximate construction of discrete PDF including each term of the PDF, which has been used in sampling the initial state, is a good option.

As we have noticed, the term $f_r(\Theta_{x_i},x_{i-1},\Theta_{x_{i-1}})\left|\cos(N_{x_{i-1}},\Theta_{x_i})\right|L(x_i,\Theta_{x_i})$ in Eq. 7 is the same with the PDF that Jensen has used [Jense95]. We simply choose the PDF

$$p[(x_i,\Theta_{x_i}),(x_{i-1},\Theta_{x_{i-1}})]\propto$$

$$f_r(\Theta_{x_i},x_{i-1},\Theta_{x_{i-1}})\left|\cos(N_{x_{i-1}},\Theta_{x_i})\right|L_{appr}(x_i,\Theta_{x_i})$$

to realize our ZVIDPT. The motivation for this is to compare our simplified version of ZVIDPT with Jensen's method, which is relatively a good Monte Carlo method [Zimme98]. With the limited length of this paper, we do not give the details here.

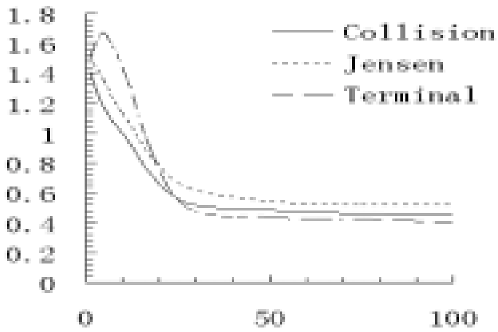
4.3 Termination of the Random Walk

The probability of termination is $q(x,\Theta_x)=L_1(x,\Theta_x)/L_{appr}(x,\Theta_x)$.
L_1 and L_{appr} here can be denoted as the color, which includes three values representing the primitive colors. The quotient is calculated using the average value of the color.

5. IMPLEMENTATION AND RESULTS

We have accomplished the new ZVIDPT algorithm in a program called THESAURUS on SGI O2 running IRIX 6.3. Test scene is an empty Cornell box and all the walls as well as the floor are diffuse. A small rectangle light source, which is placed just below the ceiling that is also diffuse, only illuminates the ceiling and little upper part of the walls. Thus the lighting of the scene is mainly indirect. All of these are much suitable for testing the convergence speed of the algorithm effectively. The reference image, which is computed using 1000 samples per pixel in the resolution 512×384 by way of ordinary stochastic ray tracing, is used to judge the magnitude of errors for the images created by different algorithms.

We have tested the ZVIDPT using both the collision estimator and the terminal estimator, as well as Jensen's importance driven path tracing by synthesizing images with 1, 2, and to 100 samples per pixel. Several important parameters used in ZVIDPT are as follows: the size of grid G_P used to sample pixel is 2×4, the size of grid G_L for sampling the middle state of random walk is 4×16, the number of photons N_p stored for the entire scene is 6891 and the number of photons N used for radiance estimation is 50. Jensen's algorithm uses the same G_L , N_p *and* N as the ZVIDPT does. In general, the larger the size of the above parameters is, the better the results are. We select all the parameters according to the fact that they can produce the reasonable results.



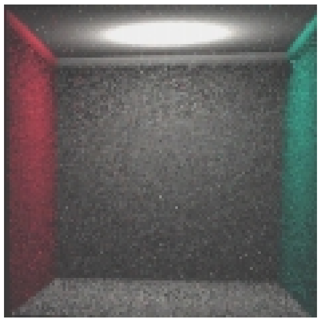
MSE error for different algorithms
Figure 1

We measure the mean square error (MSE), which is the variance of the difference between the reference image and the computed images, to quantitatively analyze the differences among several algorithms. The graph in Fig.1, which demonstrates the MSE error as a function of the number of samples for one pixel, indicates different degrees of improvement on error reduction produced by different algorithms. The ZVIDPT with the collision estimator is slightly better than Jensen's algorithm. The ZVIDPT with the terminal estimator has a relatively large initial error but can achieve faster convergence than both the ZVIDPT with the collision estimator and Jensen's method with the increased number of samples. The reduction of error is about 20% on the average. In the long run, there is no doubt that the ZVIDPT with the terminal estimator is the best one to solve the problem..

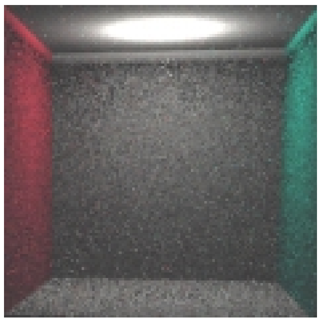
In Fig.2 we show the synthesized images with 50 samples per pixel using three different algorithms. The different level of noises can be perceived and that justifies the above results visually.

6. CONCLUSION AND FUTURE WORK

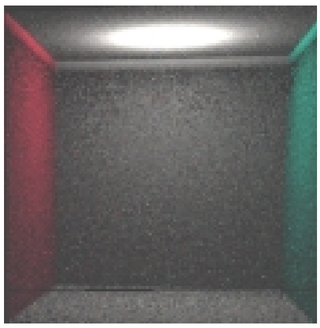
We have demonstrated that the new approach introduced by us is convenient for use and broadens the solutions to global illumination. In a word, the new approach and the previous computational way, which is widely used in current global illumination algorithms, form an integrated system to calculate the global illumination. In addition, a lot of conclusions such as plenty of unbiased estimators and zero variance importance sampling procedure only included in the new approach make it vital important.



Cornell Box: Jensen's version



Cornell Box: Collision version



Cornell Box: Terminal version
Resultant images
Figure 2

Results, which are produced from the zero variance importance driven potential tracing algorithm we have implemented, indicate the better speed up of convergence than Jensen's excellent importance driven scheme does.

In the near future, we will do:

- Methods, which are used to accurately and cheaply draw intermediate states of random walk for zero variance importance driven potential tracing, should be tried to find. This will deeply influence our algorithm.
- The procedure of zero variance importance sampling will be combined with some quasi-Monte Carlo methods to reduce errors further.

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