ROBUST BIAS-CORRECTED LEAST SQUARES FITTING OF ELLIPSES

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ABSTRACT

This paper presents a robust and accurate technique for an estimation of the best-fit ellipse going through the given set of points. The approach is based on a least squares minimization of algebraic distances of the points with a correction of the statistical bias caused during the computation. An accurate ellipse-specific solution is guaranteed even for scattered or noisy data with outliers. Although the final algorithm is iterative, it typically converges in a fraction of time needed for a true orthogonal fitting based on Eucleidan distances of points.

Keywords: ellipses, least squares, robust fitting, M-estimators, statistical bias, renormalization

1 INTRODUCTION

One of basic tasks in pattern recognition and computer vision is a fitting of geometric primitives to a set of points [Duda73, Haral93]. An application of primitive models allows reduction and simplification of the data and, consequently, faster and more reliable processing. A very important primitive is an ellipse, which, being a perspective projection of a circle, is exploited in many applications of computer vision like 3-D vision and object recognition, robot navigation, medical imaging, industrial inspections, etc.

Regarding the importance of ellipses, many different methods have been proposed for their detection and fitting. The approaches are based on various ideas (see [Zhang95] for an overview), but in principle they can be divided into two main groups: *voting/clustering* and *optimization methods*. The methods belonging to the first group (such as Hough transform [Yuen89, Yip92, Wu93], RANSAC [Rosin93, Werma95], Kalman filtering [Porri90, Rosin95], and fuzzy clustering [Davé92, Gath95]) are robust against noise and outliers among data. In addition, voting/clustering techniques are able to detect multiple primitives at once. Unfortunately, these methods are typically rather slow, memory consuming and not accurate enough.

The second group of fitting methods are based on optimization of an appropriate objective function which characterizes a goodness of a particular ellipse with respect to the given set of data points. The main advantages of these methods (such as [Books79, Taubi91, Samps92, Gande94]; see [Fitzg95] for a comparison) are their speed and accuracy, on the other hand the methods can typically fit only one primitive at time (that means that the data have to be pre-segmented before the fitting). Also the sensitivity to noise and outliers among data is higher than in the clustering methods. It should be noted that most of the optimization techniques provide general conics and the non-ellipse solutions have to be explicitly rejected. The first direct ellipse-specific fitting approach was described in [Fitzg96]. Several improvements of this method were summarized in [Halíř98].

In this paper, a modification of the ellipse-specific optimization-based fitting technique [Halíř98] is proposed. The modification copes with two most significant problems of the original method: robustness and statistical bias of the fit. The paper is organized as follows: First, the original least squares based approach is briefly described. Regarding the sensitivity to noise and outliers among data (which is common for all optimization approaches), a robust estimation technique called M-estimators is incorporated into the fitting as described in the next Section. Then, a statistical bias of the fit (caused by an application of algebraic distances during the estimation) is corrected by a renormalization procedure. Finally, the complete fitting procedure is presented together with an evaluation in several experiments. A comparison of the proposed method with the original approach concludes the whole paper.

2 DIRECT LEAST SQUARES FITTING OF ELLIPSES

This approach was firstly proposed in [Fitzg96], an improved technique is available in [Halíř98]. The method works on segmented data, i.e. all the data points are assumed to belong to one ellipse. In this section, we provide only a brief overview of the method. Further details and comparisons with another fitting approaches can be found in the given papers.

An ellipse is a special case of a general conic which can be described by an implicit second order polynomial

$$F(x,y) = ax^2 + bxy + cy^2 + dx + ey + f = 0 \quad (1)$$

with an ellipse-specific constraint

$$b^2 - 4ac < 0 \ , \quad (2)$$

where a, b, c, d, e, f are coefficients of the ellipse and (x, y) are coordinates of points lying on it. The polynomial $F(x, y)$ is called an *algebraic distance* of the point (x, y) to the given conic. By introducing vectors

$$\begin{aligned} \mathbf{a} &= [a, b, c, d, e, f]^{\mathrm{T}} \ \text{and} \\ \mathbf{x} &= [x^2, xy, y^2, x, y, 1] \ , \end{aligned} \quad (3)$$

it can be rewritten to the vector form

$$F(x,y) = F_{\mathbf{a}}(\mathbf{x}) = \mathbf{x} \cdot \mathbf{a} \ . \quad (4)$$

The fitting of an ellipse with coefficients $\mathbf{a}$ to a set of points (x_i, y_i), $i = 1, \dots, N$ can be approached by minimizing the sum of squared algebraic distances of the points to the ellipse, giving

$$\begin{aligned} \min_{\mathbf{a}} \sum_{i=1}^{N} F(x_i, y_i)^2 &= \min_{\mathbf{a}} \sum_{i=1}^{N} F_{\mathbf{a}}^2(\mathbf{x_i}) \\ &= \min_{\mathbf{a}} \sum_{i=1}^{N} (\mathbf{x_i} \cdot \mathbf{a})^2 \end{aligned} \quad (5)$$

subject to the ellipse-specific constraint Eq. 2. Because the coefficients of the ellipse $\mathbf{a}$ can be arbitrarily scaled, the constraint can be changed into

$$4ac - b^2 = 1 \ . \quad (6)$$

The constrained least squares fitting problem Eq. 5 with the constraint Eq. 6 can be expressed in a matrix form as

$$\min_{\mathbf{a}} \|\mathbf{D}\mathbf{a}\|^2 \quad \text{subject to} \quad \mathbf{a}^{\mathrm{T}}\mathbf{C}\mathbf{a} = 1 \ , \quad (7)$$

where $\mathbf{D}$ and $\mathbf{C}$ are appropriate *design* and *constraint matrices* of the sizes N×6 and 6×6, respectively.

After applying Lagrange multipliers and block decomposition of matrices, the constrained minimization problem Eq. 7 can be reformulated into the following set of equations:

$$\begin{aligned} \mathbf{M}\mathbf{a}_1 &= \lambda \mathbf{a}_1 \\ \mathbf{a}_1^{\mathrm{T}} \mathbf{C}_1 \mathbf{a}_1 &= 1 \\ \mathbf{a}_2 &= -\mathbf{S}_3^{-1} \mathbf{S}_2^{\mathrm{T}} \mathbf{a}_1 \ , \end{aligned} \quad (8)$$

where λ is a Lagrange multiplier, $\mathbf{M}$ is a *reduced scatter matrix* of the size 3×3,

$$\mathbf{M} = \mathbf{C}_1^{-1} (\mathbf{S}_1 - \mathbf{S}_2 \mathbf{S}_3^{-1} \mathbf{S}_2^{\mathrm{T}}) \quad (9)$$

and $\mathbf{C}_1$ is a *reduced constraint matrix*,

$$\mathbf{C}_1 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix} \ . \quad (10)$$

The vector of the ellipse coefficients $\mathbf{a}$ is then created from its two parts $\mathbf{a}_1$ and $\mathbf{a}_2$ as follows:

$$\mathbf{a} = \begin{pmatrix} \mathbf{a}_1 \\ \hline \mathbf{a}_2 \end{pmatrix} \ \text{where} \ \mathbf{a}_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \ \text{and} \ \mathbf{a}_2 = \begin{pmatrix} d \\ e \\ f \end{pmatrix} \quad (11)$$

The matrices $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3$ are parts of the original scatter matrix $\mathbf{S}$ defined as

$$\mathbf{S} = \left(\begin{array}{c|c} \mathbf{S}_1 & \mathbf{S}_2 \\ \hline \mathbf{S}_2^{\mathrm{T}} & \mathbf{S}_3 \end{array} \right) \quad \text{where} \quad \begin{aligned} \mathbf{S}_1 &= \mathbf{D}_1^{\mathrm{T}} \mathbf{D}_1 \\ \mathbf{S}_2 &= \mathbf{D}_1^{\mathrm{T}} \mathbf{D}_2 \\ \mathbf{S}_3 &= \mathbf{D}_2^{\mathrm{T}} \mathbf{D}_2 \end{aligned} \ . \quad (12)$$

Finally, the matrices $\mathbf{D}_1$ and $\mathbf{D}_2$ are quadratic and linear parts, respectively, of the original design matrix $\mathbf{D} = (\mathbf{D}_1 \mid \mathbf{D}_2)$:

$$\mathbf{D}_1 = \begin{pmatrix} x_1^2 & x_1 y_1 & y_1^2 \\ \vdots & \vdots & \vdots \\ x_i^2 & x_i y_i & y_i^2 \\ \vdots & \vdots & \vdots \\ x_N^2 & x_N y_N & y_N^2 \end{pmatrix} \quad (13)$$

and

$$\mathbf{D}_2 = \begin{pmatrix} x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots \\ x_i & y_i & 1 \\ \vdots & \vdots & \vdots \\ x_N & y_N & 1 \end{pmatrix} \ . \quad (14)$$

The system Eq. 8 (which is equivalent to the constrained minimization problem Eq. 7) is ready to be solved directly by the eigenvectors computation. It can be shown that

$$||\mathbf{Da}||^2 = \lambda \ , \tag{15}$$

thus the optimal solution of the fitting problem can be found as the eigenvector $\mathbf{a}_1$ corresponding to the minimal non-negative eigenvalue λ of the matrix $\mathbf{M}$ (Eq. 9). The appropriate vector of coefficients $\mathbf{a}$ (Eq. 11) then represents the best-fit ellipse for the given set of points.

3 ROBUST FITTING BY M-ESTIMATORS

The parameters of the fitted ellipse are estimated by a direct least squares minimization. This type of optimization approaches is known to be sensitive to outliers. As pointed out in [Barne84], with a least squares technique, *even one or two outliers in a large set can wreak havoc*! Regarding that, statisticians have developed various sorts of robust estimators. The most relevant class for our task are so called *M-estimators* [Beck77].

Mathematical theory of M-estimators is available in many books and papers, see for example [Rey83] or [Meer91]. The basic idea of robust estimation is to reduce the influence of outliers by replacing the squared residuals r_i^2 in the standard least squares approach

$$\min \sum_{i=1}^{N} r_i^2 \tag{16}$$

by another less increasing function ρ of the residuals, yielding

$$\min \sum_{i=1}^{N} \rho(r_i) \ . \tag{17}$$

Instead of solving Eq. 17 directly, the minimization can be implemented as an iterative reweighted process

$$\min \sum_{i=1}^{N} w(r_i^{(k-1)})\, r_i^2 \ , \tag{18}$$

where w is so called *weight function* and the superscript $^{(k)}$ indicates the iteration number. Various weight functions with different properties have been proposed in a literature, the ones suitable for parameter estimation are discussed for example in [Beck77] or [Huber81]. A very popular among statisticians is a *Huber's function*:

$$w(r_i) = \begin{cases} 1 & \text{if} \quad |r_i| \le c\hat{\sigma} \\ \dfrac{c\hat{\sigma}}{|r_i|} & \text{otherwise} \end{cases} \ , \tag{19}$$

where $\hat{\sigma}$ is a robust standard deviation of residual errors r_i and c is a tuning constant with the value $c = 1.345$. Another widely used weight function is so called *Tukey's biweight*:

$$w(r_i) = \begin{cases} \left[1 - \left(\dfrac{|r_i|}{c\hat{\sigma}}\right)^2\right]^2 & \text{if} \quad |r_i| \le c\hat{\sigma} \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

with the tuning constant $c = 4.6851$. The robust standard deviation $\hat{\sigma}$ of residual errors r_i can be estimated as [Barne84]

$$\hat{\sigma} = 1.4826 \operatorname*{median}_i |r_i| \ . \tag{21}$$

An application of the M-estimators for the robust determination of the coefficients of the best-fit ellipse is straightforward and it leads to the following algorithm:

1. Compute an initial estimate of the coefficients $\mathbf{a}$ by solving the system Eq. 8 as described in the previous Section.
2. Set the residua r_i as the algebraic distances of the points $\mathbf{x}_i$ from the currently estimated ellipse $\mathbf{a}$:

   $$r_i = F_{\mathbf{a}}(\mathbf{x}_i) = \mathbf{x}_i \cdot \mathbf{a} \ . \tag{22}$$

3. Estimate the robust standard deviation $\hat{\sigma}$ of the residua r_i (for example by Eq. 21).
4. Compute weights $w(r_i)$ from the residua r_i by the appropriate weight function w (such as Eq. 19 or Eq. 20).
5. Estimate a new set of the coefficients $\tilde{\mathbf{a}}$ which minimizes the weighted least squares problem Eq. 18.
6. Convergence check: if $||\tilde{\mathbf{a}} - \mathbf{a}|| > \varepsilon$, set $\mathbf{a} = \tilde{\mathbf{a}}$ and go back to the step 2.

The appropriate coefficients $\tilde{\mathbf{a}}$ of the weighted least squares problem Eq. 18 (step 5 of the algorithm) can be computed by the same approach as described in the previous Section, only the design matrices $\mathbf{D}_1$ (Eq. 13) and $\mathbf{D}_2$ (Eq. 14) have to be weighted by the corresponding weights w_i before the computation as follows:

$$\begin{aligned} \tilde{\mathbf{D}}_1 &= \mathbf{W} \cdot \mathbf{D}_1 \\ \tilde{\mathbf{D}}_2 &= \mathbf{W} \cdot \mathbf{D}_2 \ , \end{aligned} \tag{23}$$

where the weighting matrix $\mathbf{W}$ of the size N×N is a diagonal matrix with the squared roots of the weights w_i on the diagonal,

$$\mathbf{W} = diag(\sqrt{w_1}, \ldots, \sqrt{w_N}) \ . \tag{24}$$

4 BIAS CORRECTION BY RENORMALIZATION

In our fitting approach, the least squares minimization is performed on the algebraic distances of the points from the ellipses as can be seen in Eq. 5. The algebraic distance of the point $P_i = (x_i, y_i)$ from the ellipse E described by Eq. 1 is given as the value of the appropriate polynomial:

$$d_{alg}(P_i, E) = ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f \quad (25)$$

The application of the algebraic distances allows the formulation of the fitting task as the linear system with a closed-form solution. On the other hand, such approach has no justification from either physical or statistical viewpoint. For the "proper" fitting, Eucleidan distances have to be used instead, yielding the following minimization problem (so called *orthogonal distance fitting*):

$$\min_E \sum_{i=1}^{N} d_{geom}^2(P_i, E) \ , \quad (26)$$

where $d_{geom}(P_i, E)$ is the smallest Eucleidan distance between the point P_i and the ellipse E. Despite the excellent results, the orthogonal distance fitting approach is not widely used. The basic problem is that the computation of the Eucleidan distances d_{geom} is fairly complicated (fourth order polynomials) and that the coefficients of the optimal ellipse have to be estimated in a constrained non-linear optimization process, because no closed-form solution is known. Such estimation is slow and it could eventually fail if the initial guess of the coefficients is not accurate enough.

The least squares minimization (Eq. 5) with algebraic distances (Eq. 25) introduces so called *high curvature bias*: A point at the high curvature sections contributes less to the ellipse fitting than a point having the same amount of noise but at the low curvature sections [Kanat94]. Consequently, all fitting methods based on algebraic distances tend to produce smaller ellipses as they should be. The distortion of the provided ellipses is significant especially when the input data cover only a small section of the fitted ellipse.

In order to cope with the high curvature bias, Kanatani [Kanat93, Kanat94] proposed so called *renormalization procedure* which should be applied during the fitting of conics. The basic idea of the renormalization is that if a priori all data points are corrupted by the same amount of noise, they should contribute to the fitting equally. This can be done by a "correction" of the parameters used in the least squares fitting before the computation itself. The correction terms depend on the (typically unknown) amount of noise and also on the coefficients of the fitted conic, which leads to an iterative estimation process in which the appropriate noise distribution is gradually modeled.

The original Kanatani's renormalization approach for bias corrected fitting of conics was further improved by Zhang in [Zhang95, chapter 7]. Here we propose a modification of the original Zhang's method for the ellipse-specific fitting technique described in the beginning of this paper.

Assume that all input points $P_i = (x_i, y_i)$ are perturbed by Gaussian noise ΔP_i, then the matrices $\mathbf{M}$ (Eq. 9) and $\mathbf{S}_k$ (Eq. 12) are perturbed accordingly:

$$\begin{aligned} \mathbf{M} &= \overline{\mathbf{M}} + \Delta\mathbf{M} \\ \mathbf{S}_k &= \overline{\mathbf{S}}_k + \Delta\mathbf{S}_k \quad k = 1,2,3 \ , \end{aligned} \quad (27)$$

where $\overline{\mathbf{M}}$ and $\overline{\mathbf{S}}_k$ are the (unknown) unperturbed matrices. Due to the perturbation of the matrices, the coefficients $\mathbf{a}$ of the fitted ellipse estimated from the system Eq. 8 are distorted.

In order to obtain unbiased solution of the fitting task, the ellipse coefficients should be computed from the appropriate unperturbed matrices $\overline{\mathbf{S}}_k$ and $\overline{\mathbf{M}}$. Unfortunately, these matrices are not a priori known. For their estimation, the expectations of $\Delta\mathbf{M}$ and $\Delta\mathbf{S}_k$ have to be known. According to the derivation given in [Zhang95, chapter 7], these unperturbed matrices can be expressed as follows:

$$\overline{\mathbf{M}} = \mathbf{C}_1^{-1}\left(\overline{\mathbf{S}}_1 - \overline{\mathbf{S}}_2\overline{\mathbf{S}}_3^{-1}\overline{\mathbf{S}}_2^{\mathrm{T}}\right) \quad (28)$$

(compare with Eq. 9) and

$$\overline{\mathbf{S}}_k = \mathbf{S}_k - \sigma^2 \cdot \Delta\mathbf{S}_k \quad k = 1,2,3 \ , \quad (29)$$

where σ^2 is the variance of the input noise and the correction matrices $\Delta\mathbf{S}_k$ have the following forms:

$$\Delta\mathbf{S}_1 = \begin{pmatrix} 6 \cdot S_{x^2} & 3 \cdot S_{xy} & S_{x^2} + S_{y^2} \\ 3 \cdot S_{xy} & S_{x^2} + S_{y^2} & 3 \cdot S_{xy} \\ S_{x^2} + S_{y^2} & 3 \cdot S_{xy} & 6 \cdot S_{y^2} \end{pmatrix} \quad (30)$$

$$\Delta\mathbf{S}_2 = \begin{pmatrix} 3 \cdot S_x & S_y & S_1 \\ S_y & S_x & 0 \\ S_x & 3 \cdot S_y & S_1 \end{pmatrix} \quad (31)$$

$$\Delta\mathbf{S}_3 = \begin{pmatrix} S_1 & 0 & 0 \\ 0 & S_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \quad (32)$$

where the operator S denotes the sum

$$S_{x^ay^b} = \sum_{i=1}^{N} x_i^a y_i^b \ . \quad (33)$$

Because the variance σ^2 of the input noise is typically not a priori known, it is estimated iteratively, which leads to the following renormalization algorithm (consult the original Zhang's paper for more details):

1. Set $\sigma^2 = 0$ (i.e. assume no noise for the first estimation).
2. Construct the matrices required for the estimation: $\mathbf{D}_1$ (Eq. 13), $\mathbf{D}_2$ (Eq. 14), $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3$ (Eq. 12), $\Delta\mathbf{S}_1$ (Eq. 30), $\Delta\mathbf{S}_2$ (Eq. 31), $\Delta\mathbf{S}_3$ (Eq. 32), $\overline{\mathbf{S}}_1$, $\overline{\mathbf{S}}_2$ and $\overline{\mathbf{S}}_3$ (Eq. 29) and finally $\overline{\mathbf{M}}$ (Eq. 28).
3. Estimate the coefficients $\mathbf{a}$ by solving the system Eq. 8 and Eq. 11.
4. Update the estimate of the noise variance $\sigma^2 \leftarrow \sigma^2 + \lambda/\delta$, where λ is the eigenvalue of the matrix $\mathbf{M}$ corresponding to the eigenvector $\mathbf{a}_1$ (see Eq. 8) and
$$\delta = \mathbf{a}^{\mathrm{T}} \cdot \left(\begin{array}{c|c} \Delta\mathbf{S}_1 & \Delta\mathbf{S}_2 \\ \hline \Delta\mathbf{S}_2^{\mathrm{T}} & \Delta\mathbf{S}_3 \end{array} \right) \cdot \mathbf{a} \tag{34}$$
is the appropriate correction term.
5. Convergence check: finish if the update has converged (i.e. if $\|\lambda/\delta\| < \varepsilon$), otherwise go back to the step 2.

The final coefficients $\mathbf{a}$ represent the best bias-corrected elliptical fit through the given points. If it is desirable, the appropriate value of σ^2 can be used as an estimate of the variance of the noise among the data.

It should be noted that our renormalization procedure does not scale the points by their gradients as was proposed in the original Zhang's approach. It was practically verified that the application of the gradients typically does not improve the estimation significantly and it only slows down the computation. Moreover, by ignoring the gradients, the proposed renormalization procedure can be easily combined together with the previously described technique of M-estimators.

5 COMPLETE FITTING PROCEDURE

In order to make the fitting of ellipses both robust and bias-corrected, the two approaches proposed in the previous Sections have to be combined together. With respect to the similar characteristics of both M-estimators and the renormalization technique, the combination is pretty straightforward. The final robust bias-corrected least squares fitting of ellipses based on algebraic distances can be performed as follows:

1. Initialization: Set $\sigma^2 = 0$ and $w_i = 1$ for all $i = 1, \ldots, N$ (i.e. assume no noise and equal weights for the first estimation); also create the design matrices $\mathbf{D}_1$ (Eq. 13) and $\mathbf{D}_2$ (Eq. 14) from the coordinates of the points.
2. Create the weighted design matrices $\tilde{\mathbf{D}}_1$ and $\tilde{\mathbf{D}}_2$ by weighting all rows of the matrices $\mathbf{D}_1$ and $\mathbf{D}_2$ by the appropriate weights $\sqrt{w_i}$ (see Eq. 23 and Eq. 24).
3. Create the scatter matrices $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3$ (Eq. 12), but use weighted $\tilde{\mathbf{D}}_1$ and $\tilde{\mathbf{D}}_2$ instead of the original $\mathbf{D}_1$ and $\mathbf{D}_2$.
4. Estimate the corrections $\Delta\mathbf{S}_1$ (Eq. 30), $\Delta\mathbf{S}_2$ (Eq. 31), $\Delta\mathbf{S}_3$ (Eq. 32), but use the weighted variant $\tilde{S}$ of the original operator S (Eq. 33) defined as:
$$\tilde{S}_{x^a y^b} = \sum_{i=1}^{N} w_i x_i^a y_i^b \ . \tag{35}$$
Note that all the corrections $\Delta\mathbf{S}_k$ can be created directly from the elements of the scatter matrices $\mathbf{S}_k$ without any additional computation.
5. Estimate the unperturbed matrices $\overline{\mathbf{S}}_1$, $\overline{\mathbf{S}}_2$ and $\overline{\mathbf{S}}_3$ (Eq. 29) and $\overline{\mathbf{M}}$ (Eq. 28).
6. Estimate the ellipse coefficients $\mathbf{a}$ by solving the system Eq. 8 and Eq. 11.
7. Update the estimate of the noise variance $\sigma^2 \leftarrow \sigma^2 + \lambda/\delta$, where λ is the eigenvalue of the matrix $\mathbf{M}$ corresponding to the eigenvector $\mathbf{a}_1$ (see Eq. 8) and the correction term δ is given by Eq. 34.
8. Set the residua r_i as the algebraic distances (Eq. 25) of points $P_i = (x_i, y_i)$ from the currently estimated ellipse E defined by the coefficients $\mathbf{a}$ as given in Eq. 1:
$$r_i = d_{alg}(P_i, E) \ . \tag{36}$$
9. Estimate the robust standard deviation $\hat{\sigma}$ of the residua r_i (for example by Eq. 21).
10. Compute new weights $\tilde{w}_i$ by the appropriate weight function w (such as Eq. 19 or Eq. 20):
$$\tilde{w}_i = w(r_i) \ . \tag{37}$$
11. Convergence check: finish if the estimation process has converged, otherwise set $w_i = \tilde{w}_i$ and go back to the step 2.

It can be seen that despite its relative complexity, the final fitting algorithm only combines steps from the previous two techniques into one loop. The only problem is the definition of the criterion of convergence. It the loop, the following three parameters are changed:

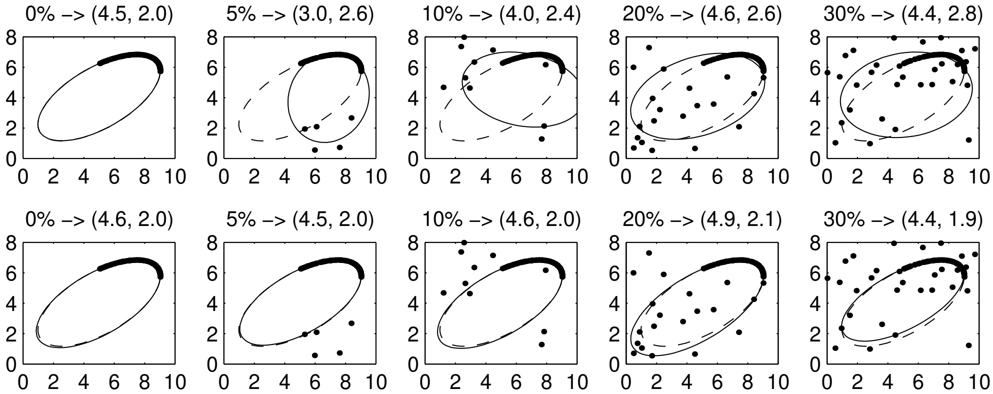

Figure 1: A demonstration of the robustness of the proposed fitting technique: A synthetic set of points representing the same ellipse (drawn by a dashed line in each figure) with an increasing number of outliers was fitted by the original approach [Halíř98] (the first row) and the new robust technique (the second row). Note significant improvements of the estimated ellipses. See text for more details.

- the estimate of the noise variance σ^2 (due to the renormalization process),
- the weights of the points w_i (with respect to the residua r_i as driven by the technique of M-estimators), and, finally,
- the coefficients of the fitted ellipse $\mathbf{a}$.

From a wide range of possible criteria of convergence, the ones based on weights such as

$$\text{finish if} \quad \|\tilde{w} - w\| < \varepsilon \tag{38}$$

or

$$\text{finish if} \quad \max_i |\tilde{w}_i - w_i| < \varepsilon \tag{39}$$

were evaluated as the most efficient estimators of the number of iterations needed for fast but accurate determination of the appropriate ellipse coefficients. Such definition of the convergence criterion resulted mainly from the fact that the renormalization process (i.e. the proper estimate of the parameter σ^2) typically requires less number of iterations (3–6) than the stabilization of the weights w_i performed by the method of M-estimators (5–15 depending on the number of outliers among data).

6 EXPERIMENTAL RESULTS

The complete fitting technique was implemented in MATLAB [Mathw84] directly as described in the previous Section. The final program was evaluated in many experiments. Being an improvement of the original fitting method [Halíř98], our approach preserves its favorable properties such as unambiguous and stable results, guaranteed ellipse specific solution and invariance of the solution to an affine transformation of the data points. In addition, the novel technique brings robustness against outliers and noise among data and it also corrects the statistical bias of the provided ellipses caused by the application of algebraic distances during the estimation.

The properties of the fitting algorithm were verified on synthetic data sets similarly as in the original papers [Fitzg96] and [Halíř98]. Some results of these experiments are presented in Fig. 1 and Fig. 2.

Fig. 1 illustrates the robustness of the proposed method against outliers among data. In this experiment, a synthetic set of 100 points which covers a section from 345 to 75 degrees of an ellipse with center (5, 4), semiaxes (4.5, 2) and tilt 30 degrees was created first. All the points were blurred by a small amount of additional Gaussian noise and an increasing number of points were completely replaced by randomly generated outliers. The final set was fitted by an ellipse. The results of the original method [Halíř98] are depicted in the first row, whereas the second row presents the ellipses obtained by the robust approach proposed in this paper. In each figure, the ideal ellipse is drawn by a dashed line and the appropriate number of outliers and lengths of the semiaxes of the fitted ellipse are noted. A significant improvements of the estimates provided by the new approach are clearly visible.

The second experiment (Fig. 2) demonstrates the ability of the renormalization technique to correct the high curvature bias. The same "synthetic ellipse" as in the previous experiment was used, but now the points gradually cover smaller and

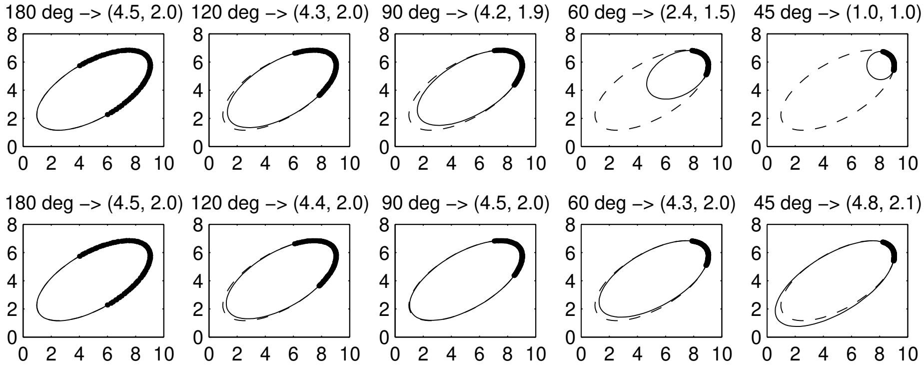

Figure 2: An example of the bias correction available in the proposed fitting technique: A synthetic set of points representing the same ellipse (drawn by a dashed line in each figure) which covers smaller and smaller section of the ellipse was fitted by the original approach [Halíř98] (the first row) and the proposed algorithm which contains a renormalization procedure (the second row). Note that even for small elliptical sections available for the fitting, the ellipse estimated by the new technique does not suffer from the high curvature bias which is easily remarkable in the results of the original approach. See text for more details.

smaller section of the ellipse starting from 180 down to 45 degrees. In the results of the original fitting method depicted in the first row, noticeable distortions of the fitted ellipse caused by the high curvature bias can be detected especially for the sections smaller than 90 degrees. The renormalization procedure included in the improved technique proposed in this paper successfully correct the bias as can be verified in the second row.

7 CONCLUSION AND OUTLOOK

This paper is devoted to an estimation of the best-fit ellipse through the given set of points. In particular, a least squares fitting based on algebraic distances of the points is investigated. Our approach was inspired by a novel direct ellipse-specific technique which was proposed in [Fitzg96] and improved in [Halíř98]. Unfortunately, the original method suffers from two problems: sensitivity to noise and outliers among data and distortion of the provided estimates by so called high curvature bias. To improve the robustness of the fitting, we propose an application of a robust estimation technique called M-estimators. In addition, the new approach also exploits a renormalization procedure which is able to cope with the biased estimates. Both improvements were combined together yielding the complete fitting technique described in this paper.

The proposed fitting algorithm is iterative, which is directed by the needs of both the M-estimators and the renormalization procedure to dynamically change their parameters with respect to the actual estimate of the ellipse. However, the method converges in a small number of iterations (5–15) which brings reasonably fast estimation (about an order faster than a true orthogonal fitting based on Eucleidan distances of points). Being an improvement of the original fitting technique, the proposed approach preserves its favorable properties such as numerical stability, unambiguous results and guaranteed ellipse specific solution. In addition, our method brings robustness against outliers and noise among data and it also corrects the statistical bias of the provided estimates. Significant improvements of the fitted ellipses (notable especially for small elliptical sections with outliers) were verified in many experiments.

Because only an ellipse in the "base position" (i.e. with no tilt) or even a circle should be fitted in many applications, a special modification of the proposed fitting algorithm for such tasks is currently investigated. On the other hand, techniques leading to the ability of our method to simultaneously detect multiple ellipses (such as clustering or so called minimum description length principle) should be considered too.

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