A DEFORMATION MODEL OF THIN FLEXIBLE SURFACES

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ABSTRACT

In engineering, it is often needed to determine the deformed shape of thin flexible surfaces, such as fabrics, paper, etc. under the action of applied loads. In this paper, a variational formulation was developed for such cases. It serves as the foundation for developing the numerical solution scheme. Through the variational formulation, the governing differential equation of motion was derived. The widely used Terzopoulos model is shown to be an approximated solution. The constitutive relationship for thin flexible surfaces was established. Finally the draping effect of a skirt was simulated based on this model. The formulation is suitable for most large deformation problems and is extendable to cover the non-linear materials.

Keywords: Variational formulation; Flexible surface; Differential equation of motion; Finite difference solution; Deformation model

INTRODUCTION

In recent years, cloth animation has become an important subject in computer animation. It has the potential to revolutionize the traditional 2D process of garment design, and allow users to visualize the fitness of garments before they are actually manufactured. Whereas, to simulate the deformation behavior of such thin flexible surfaces as fabrics, many unusual characteristics of deformation, such as large rotation and small strain, non-linear stress-strain relation, two-dimensional parametric surfaces, etc., should be considered in the formulation.

There existed three main methods to simulate the deformation behavior, such as draping, of thin flexible surfaces. The first approach is the finite element models, e.g. [Aono90], [Kang95]. They used classical plate or shell elements to model the flexible surface. The associated numerical analysis was complicated and computationally expensive for such large deformation problems coupled with dynamic contact boundary condition and non-linear material behaviors. The second approach is the particle-system model [Reyno87], [Breen94], [Eberh96] in which the cloth was represented by many interacting particles. By simulating the trajectories of every particle, one can obtain the deformation pattern of the cloth. It is a even more time-consuming process and the equivalence between the cloth with the set of interacting particles needs to be improved. The third method , which can be called as the deformation surface model , was mainly developed by Terzopoulos and his colleagues [Terzo88], [Qin95], [Qin97]. It has been widely used in draping simulations of cloth [Celni91]; [Guduk94]; [Carig92]. It is a well-known fact that the deformation surface model developed by Terzopoulos et al. is an approximate model, and it is not easy to incorporate the cloth properties, such as Young's modulus, Poisson's ratio and bending rigidity into the model. In this paper, a two-dimensional parametric variational formulation was developed to derive the exact governing differential equation for the flexible surface with the coefficients of the equation related directly to the material's constants. The derivation corrects the bending term in Terzopoulos et al' equation. The draping effect of a skirt was used to demonstrate the result of the finite difference solution based on the formulation.

VARIATIONAL FORMULATIONS

The position of any point on a thin surface in Euclidean space can be described by

$$\vec{x}(u,t) = [x_1(u,t), x_2(u,t), x_3(u,t)] \quad (1)$$

where $u(u_1, u_2)$ represents the surface's local coordinate in the parametric space. It is assumed that the external forces acting on the moving surface are: (1) the inertial force $\vec{f}_1 = -\rho \frac{\partial^2 \vec{x}}{\partial t^2}$, where ρ is the mass density, in the direction opposite to the acceleration; (2) the velocity-dependent damping force, $\vec{f}_2 = -\gamma \frac{\partial \vec{x}}{\partial t}$, in the direction opposite to the velocity; (3) the applied force

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$\vec{p}$, such as gravity, or the restricted forces acting on the surface. If $\vec{x}(u,t)$ is denoted as the equilibrium position at time t, according to the principle of virtual work [Pilke94], we know that the sum of the external virtual work and the internal virtual work is zero for virtual displacement $\delta\vec{x}$ that satisfies the kinematic equations and kinematic boundary conditions. The external virtual work can be written in the form as

$$\delta W_e = \iint_\Omega \left(-\rho\frac{\partial^2\vec{x}}{\partial t^2} - \gamma\frac{\partial\vec{x}}{\partial t} + \vec{p}\right)\cdot\delta\vec{x}\,du_1 du_2 \tag{2}$$

where Ω is the domain occupied by the surface.

The internal virtual work is given by [Axelr87]

$$\delta W_i = -\delta\iint F(G_{ij}, B_{ij})\,du_1 du_2 \tag{3}$$

where $F(G_{ij}, B_{ij})$ is the strain energy per unit area, and G_{ij}, B_{ij} are the first and second fundamental forms of the surface, which are given by

$$G_{ij} = \frac{\partial\vec{x}}{\partial u_i}\cdot\frac{\partial\vec{x}}{\partial u_j},\ B_{ij} = \vec{n}\cdot\frac{\partial^2\vec{x}}{\partial u_i\partial u_j} \tag{4}$$

where $\vec{n}$ is the unit surface normal which is given by

$$\vec{n} = \frac{\dfrac{\partial\vec{x}}{\partial u_1}\times\dfrac{\partial\vec{x}}{\partial u_2}}{\left|\dfrac{\partial\vec{x}}{\partial u_1}\times\dfrac{\partial\vec{x}}{\partial u_2}\right|} \tag{5}$$

Expressing the strain energy density $F(G_{ij}, B_{ij})$ as a function of G_{ij}, B_{ij} is sufficiently general for thin flexible surface deformation. Therefore

$$\delta W_i + \delta W_e = 0 \tag{6}$$

from which one can derive the governing differential equations for determining the equilibrium positions. The variation of the strain energy, equation (3), can be written in the form as

$$\delta\iint_\Omega F(G_{ij}, B_{ij})\,du_1 du_2 = \iint_\Omega \left(\frac{\partial F}{\partial G_{ij}}\delta G_{ij} + \frac{\partial F}{\partial B_{ij}}\delta B_{ij}\right)du_1 du_1 \tag{7}$$

and using Green's theorem,

$$\begin{aligned}\iint_\Omega \frac{\partial F}{\partial G_{ij}}\delta G_{ij}\,du_1 du_2 &= \oint_\Gamma \frac{\partial F}{\partial G_{ij}}\left(\frac{\partial\vec{x}}{\partial u_i}a_j + \frac{\partial\vec{x}}{\partial u_j}a_i\right)\delta\vec{x}\,dl \\ &- \iint_\Omega \left[\frac{\partial}{\partial u_j}\left(\frac{\partial F}{\partial G_{ij}}\cdot\frac{\partial\vec{x}}{\partial u_i}\right) + \frac{\partial}{\partial u_i}\left(\frac{\partial F}{\partial G_{ij}}\cdot\frac{\partial\vec{x}}{\partial u_j}\right)\right]\delta\vec{x}\,du_1 du_2\end{aligned} \tag{8}$$

where Γ is the contour of the surface, and a_i is the projection of the normal on u_i axis. The transformation of the second term in equation (7) is somewhat more lengthy. It can be written in the form as

$$\iint_\Omega \frac{\partial F}{\partial B_{ij}}\delta B_{ij}\,du_1 du_2 = \iint_\Omega \frac{\partial F}{\partial B_{ij}}\left(\delta\vec{n}\cdot\frac{\partial^2\vec{x}}{\partial u_i\partial u_j} + \vec{n}\cdot\frac{\partial^2\delta\vec{x}}{\partial u_i\partial u_j}\right)du_1 du_2 \tag{9}$$

The second term in equation (9) can be written in the form as

$$\begin{aligned}\iint_\Omega \frac{\partial F}{\partial B_{ij}}\left(\vec{n}\cdot\frac{\partial^2\delta\vec{x}}{\partial u_i\partial u_j}\right)du_1 du_2 &= \oint_\Gamma \frac{\partial F}{\partial B_{ij}}\vec{n}\cdot\frac{\partial\delta\vec{x}}{\partial u_i}a_j\,dl - \oint_\Gamma \frac{\partial}{\partial u_j}\left(\frac{\partial F}{\partial B_{ij}}\vec{n}\right)\cdot\delta\vec{x}\,a_i\,dl \\ &+ \iint_\Omega \frac{\partial^2}{\partial u_i\partial u_j}\left(\frac{\partial F}{\partial B_{ij}}\vec{n}\right)\cdot\delta\vec{x}\,du_1 du_2\end{aligned} \tag{10}$$

If we write the normal vector $\vec{n}$ given by equation (5) in the following form

$$\vec{n} = \frac{1}{N}(n_x\vec{i} + n_y\vec{j} + n_z\vec{k}) \tag{11}$$

where

$$n_x = \frac{\partial x_2}{\partial u_1}\cdot\frac{\partial x_3}{\partial u_2} - \frac{\partial x_3}{\partial u_1}\cdot\frac{\partial x_2}{\partial u_2} \tag{12}$$

$$n_y = \frac{\partial x_3}{\partial u_1}\cdot\frac{\partial x_1}{\partial u_2} - \frac{\partial x_1}{\partial u_1}\cdot\frac{\partial x_3}{\partial u_2} \tag{13}$$

$$n_z = \frac{\partial x_1}{\partial u_1}\cdot\frac{\partial x_2}{\partial u_2} - \frac{\partial x_2}{\partial u_1}\cdot\frac{\partial x_1}{\partial u_2} \tag{14}$$

$$N = (n_x^2 + n_y^2 + n_z^2)^{1/2} \tag{15}$$

one obtains

$$\delta\vec{n}\cdot\frac{\partial^2\vec{x}}{\partial u_i\partial u_j} = A_{11}\frac{\partial\delta x_1}{\partial u_2} + A_{22}\frac{\partial\delta x_1}{\partial u_1} + B_{11}\frac{\partial\delta x_2}{\partial u_2} + B_{22}\frac{\partial\delta x_2}{\partial u_1} + D_{11}\frac{\partial\delta x_3}{\partial u_2} + D_{22}\frac{\partial\delta x_3}{\partial u_1} \tag{16}$$

where

$$\begin{aligned} a_{11} &= n_y\frac{\partial x_3}{\partial u_1} - n_z\frac{\partial x_2}{\partial u_1},\ a_{22} = n_z\frac{\partial x_2}{\partial u_2} - n_y\frac{\partial x_3}{\partial u_2} \\ b_{11} &= n_z\frac{\partial x_1}{\partial u_1} - n_x\frac{\partial x_3}{\partial u_1},\ b_{22} = n_x\frac{\partial x_3}{\partial u_2} - n_z\frac{\partial x_1}{\partial u_2} \\ c_{11} &= n_x\frac{\partial x_2}{\partial u_1} - n_y\frac{\partial x_1}{\partial u_1},\ c_{22} = n_y\frac{\partial x_1}{\partial u_2} - n_x\frac{\partial x_2}{\partial u_2}\end{aligned} \tag{17}$$

and

$$A_{11} = -a_{11}\frac{n_x}{N^2}\frac{\partial^2 x_1}{\partial u_i \partial u_j} + \left(\frac{\partial x_3}{\partial u_1} - a_{11}\frac{n_y}{N^2}\right)\frac{\partial^2 x_2}{\partial u_i \partial u_j} - \left(\frac{\partial x_2}{\partial u_2} + a_{11}\frac{n_z}{N^2}\right)\frac{\partial^2 x_3}{\partial u_i \partial u_j}$$
$$A_{22} = -a_{22}\frac{n_x}{N^2}\frac{\partial^2 x_1}{\partial u_i \partial u_j} - \left(\frac{\partial x_3}{\partial u_2} + a_{22}\frac{n_y}{N^2}\right)\frac{\partial^2 x_2}{\partial u_i \partial u_j} + \left(\frac{\partial x_2}{\partial u_2} - a_{22}\frac{n_z}{N^2}\right)\frac{\partial^2 x_3}{\partial u_i \partial u_j}$$
$$B_{11} = -b_{11}\frac{n_y}{N^2}\frac{\partial^2 x_2}{\partial u_i \partial u_j} + \left(\frac{\partial x_1}{\partial u_1} - b_{11}\frac{n_z}{N^2}\right)\frac{\partial^2 x_3}{\partial u_i \partial u_j} - \left(\frac{\partial x_3}{\partial u_1} + b_{11}\frac{n_x}{N^2}\right)\frac{\partial^2 x_1}{\partial u_i \partial u_j}$$
$$B_{22} = -b_{22}\frac{n_y}{N^2}\frac{\partial^2 x_2}{\partial u_i \partial u_j} + \left(\frac{\partial x_3}{\partial u_2} - b_{22}\frac{n_x}{N^2}\right)\frac{\partial^2 x_1}{\partial u_i \partial u_j} - \left(\frac{\partial x_1}{\partial u_2} + b_{22}\frac{n_z}{N^2}\right)\frac{\partial^2 x_3}{\partial u_i \partial u_j}$$
$$D_{11} = -c_{11}\frac{n_z}{N^2}\frac{\partial^2 x_3}{\partial u_i \partial u_j} + \left(\frac{\partial x_2}{\partial u_1} - c_{11}\frac{n_x}{N^2}\right)\frac{\partial^2 x_1}{\partial u_i \partial u_j} - \left(\frac{\partial x_1}{\partial u_1} + c_{11}\frac{n_y}{N^2}\right)\frac{\partial^2 x_2}{\partial u_i \partial u_j}$$
$$D_{22} = -c_{22}\frac{n_z}{N^2}\frac{\partial^2 x_3}{\partial u_i \partial u_j} + \left(\frac{\partial x_1}{\partial u_2} - c_{22}\frac{n_y}{N^2}\right)\frac{\partial^2 x_2}{\partial u_i \partial u_j} - \left(\frac{\partial x_2}{\partial u_2} + c_{22}\frac{n_x}{N^2}\right)\frac{\partial^2 x_1}{\partial u_i \partial u_j} \tag{18}$$

By substituting equation (16) into equation (9), one obtains

$$\iint_\Omega \frac{\partial F}{\partial B_{ij}}\delta\vec{n}\cdot\frac{\partial^2\vec{x}}{\partial u_i \partial u_j}du_1 du_2 = \oint_\Gamma \frac{\partial F}{\partial B_{ij}}[(A_{11}a_2 + A_{22}a_1)\delta x_1 + (B_{11}a_2 + B_{22}a_1)\delta x_2 + (D_{11}a_2 + D_{22}a_1)\delta x_3]dl - \iint_\Omega \vec{H}\cdot\delta\vec{x}du_1 du_2 \tag{19}$$

where

$$\vec{H} = H_1\vec{i} + H_2\vec{j} + H_3\vec{k} \tag{20}$$

and

$$H_1 = \frac{\partial}{\partial u_2}(A_{11}\frac{\partial F}{\partial B_{ij}}) + \frac{\partial}{\partial u_1}(A_{22}\frac{\partial F}{\partial B_{ij}})$$
$$H_2 = \frac{\partial}{\partial u_2}(B_{11}\frac{\partial F}{\partial B_{ij}}) + \frac{\partial}{\partial u_1}(B_{22}\frac{\partial F}{\partial B_{ij}}) \tag{21}$$
$$H_3 = \frac{\partial}{\partial u_2}(D_{11}\frac{\partial F}{\partial B_{ij}}) + \frac{\partial}{\partial u_1}(D_{22}\frac{\partial F}{\partial B_{ij}})$$

Along the contour of the surface, if the positions are given, then $\delta\vec{x} = 0$, and all the terms in the contour integrals are zero. For the case where the external force is given along the contour, one should add the terms of external virtual work along the contour. Thus substitution of equations (8), (9) and related terms into equation (7) gives both the surface and contour integrals. Since the variation $\delta\vec{x}$ in both integrals is arbitrary, the integrals can therefore vanish only if the coefficients of $\delta\vec{x}$ are zero. Equating the coefficient in the contour integral to zero gives the boundary traction equilibrium condition. Whereas, that the coefficient in the surface integral is zero gives the following equation

$$\rho\frac{\partial^2\vec{x}}{\partial t^2} + \gamma\frac{\partial\vec{x}}{\partial t} - \frac{\partial}{\partial u_j}(\frac{\partial F}{\partial G_{ij}}\cdot\frac{\partial\vec{x}}{\partial u_i}) - \frac{\partial}{\partial u_i}(\frac{\partial F}{\partial G_{ij}}\cdot\frac{\partial\vec{x}}{\partial u_j}) + \frac{\partial}{\partial u_i \partial u_j}(\vec{n}\frac{\partial F}{\partial B_{ij}}) - \vec{H} = \vec{P} \tag{22}$$

where $\vec{H}$ is given by Eq.(20), and $\vec{P}$ is the external force acting on the surface. In Terzopoulos and subsequent works, the term $\vec{H}$ was neglected. In the following part, we will give the expression of the strain energy density for linear elastic materials, and show how to determine the constants through experiments.

THE STRAIN ENERGY DENSITY OF A LINEAR ELASTIC SURFACE

For a general linear anisotropic elastic material, the strain energy density can be written as

$$F(G_{ij}, B_{ij}) = \frac{1}{8}(G_{ij} - G_{ij}^0)C_{ijkl}^1(G_{kl} - G_{kl}^0) + (B_{ij} - B_{ij}^0)C_{ijkl}^2(B_{kl} - B_{kl}^0) \tag{23}$$

where Einstein's summation convention has been used. The associated errors is at most at relative order of $|h/R| + (h/L)^2$, where h is the thickness of the surface, R denotes the smallest principal radius of curvature, and L is a characteristic wave-length of the deformation pattern [Axelr87], and G_{ij}^0, B_{ij}^0 are the first and second fundamental forms of the surface at the undeformed state. In equation (23), the tensors $\mathbf{C^1}$ and $\mathbf{C^2}$ have the same symmetry properties as the elastic modulus tensor

$$C_{ijkl}^1 = C_{jikl}^1 = C_{ijlk}^1 = C_{klij}^1$$
$$C_{ijkl}^2 = C_{jikl}^2 = C_{ijlk}^2 = C_{klij}^2 \tag{24}$$

If the surface is made of orthotropic material with u_1 and u_2 as the principal symmetry directions, one can show that

$$C_{1111}^1, C_{2222}^1, C_{1212}^1, C_{1111}^2, C_{2222}^2, C_{1212}^2 \neq 0 \tag{25}$$

while all the other components are zero. Therefore equation (23) becomes

$$F = \frac{1}{8}C_{1111}^1(G_{11} - G_{11}^0)^2 + \frac{1}{8}C_{2222}^1(G_{22} - G_{22}^0)^2 + \frac{1}{4}C_{1212}^1(G_{12} - G_{12}^0)^2 + C_{1111}^2(B_{11} - B_{11}^0)^2 + C_{2222}^2(B_{22} - B_{22}^0)^2 + C_{1212}^2(B_{12} - B_{12}^0)^2 \tag{26}$$

Compared with Terzopoulos et al's result, one can find that the coefficients C_{ijkl}^1, C_{ijkl}^2 are the weighting functions. Whereas through comparison with the exact form, equation (22), which we have derived, it is very clear that their tension and shear terms associated with G_{ij} are correct, whereas, the bending terms associated with B_{ij} only approximate the correct expressions. It can also be found that by considering the bending terms, one cannot derive the uncoupled equations for the components of $\vec{x}$ as Terzopoulos did. If we assume that u_i and x_i have the same dimension of length, the dimension of the bending terms in Terzopoulos et al's equation is not consistent with the other terms.

In what follows, the coefficients C^1_{ijkl}, C^2_{ijkl} will be determined from suitably selected solutions corresponding to the experimental measurements. For thin flexible surfaces, such as fabrics, papers and non-wovens, the mechanical properties can be measured by using the Kawabata's Evaluation System. It can measure not only the tensile and bending properties along different directions, but also the shear modulus on the surface. Through these experimental data, we can determine the weighting functions C^1_{ijkl}, C^2_{ijkl}. For orthotropic materials, the six non-zero components can be determined as follows.

Pure Tension Along u_1 Direction

In such case, if Young's modulus is measured as E_1, the strain energy in unit area is given by

$$F = \frac{1}{2} E_1 \varepsilon_1^2 h \qquad (27)$$

where $\varepsilon_1 (= \frac{1}{2}(G_{11} - G^0_{11}))$ is the tensile strain along u_1 direction and h is the thickness of the surface. From equation (26), one can write

$$F = \frac{1}{2} C^1_{1111} \varepsilon_1^2 + \frac{1}{2} \gamma^2 C^1_{2222} \varepsilon_1^2 \qquad (28)$$

where γ is the Poisson's ratio. In deriving equation (28), we have used $\varepsilon_2 = \frac{1}{2}(G_{22} - G^0_{22}) = \gamma \varepsilon_1$. Through equations (27) and (28), one finds

$$C^1_{1111} + \gamma^2 C^1_{2222} = E_1 h \qquad (29)$$

Pure Tension Along u_2 Direction

In such case, if Young's modulus is measured as E_2 along u_2 direction, the strain energy in unit area is given by

$$F = \frac{1}{2} E_2 \varepsilon_2^2 h \qquad (30)$$

where $\varepsilon_2 (= \frac{1}{2}(G_{22} - G^0_{22}))$ is the tensile strain along u_2 direction. From equation (26), one can write

$$F = \frac{1}{2} \gamma^2 C^1_{1111} \varepsilon_2^2 + \frac{1}{2} C^1_{2222} \varepsilon_2^2 \qquad (31)$$

Therefore one can find

$$\gamma^2 C^1_{1111} + C^1_{2222} = E_2 h \qquad (32)$$

Combining equations (29) and (32), one can obtain the coefficients C^1_{1111}, C^1_{2222} as functions of material's constants E_1, E_2, γ.

Pure Shear Test

In such case, if the shear modulus on the surface denoted as G , the strain energy in unit area is given by

$$F = \frac{1}{2} G \varepsilon_{12}^2 h \qquad (33)$$

where $\varepsilon_{12} (= \frac{1}{2}(G_{12} - G^0_{12}))$ is the shear strain. From equation (26), one can write

$$F = C^1_{1212} \varepsilon_{12}^2 \qquad (34)$$

Therefore, one finds

$$C^1_{1212} = Gh \qquad (35)$$

Pure Bending Along u_1 Direction

In such case, if the bending rigidity is measured as K_1, the strain energy in unit area is given by

$$F = K_1 (B_{11} - B^0_{11})^2 \qquad (36)$$

From equation (26), one can write

$$F = C^2_{1111}(B_{11} - B^0_{11})^2 + \gamma'^2 C^2_{2222}(B_{11} - B^0_{11})^2 \qquad (37)$$

where $\gamma'(B_{11} - B^0_{11}) = B_{22} - B^0_{22}$ is the bending component along u_2 due to the bending component $(B_{11} - B^0_{11})$ along u_1. Through equations (36) and (37), one finds

$$C^2_{1111} + \gamma'^2 C^2_{2222} = K_1 \qquad (38)$$

Pure Bending Along u_2 Direction

In such case, if the bending rigidity is measured as K_2, the strain energy in unit area is given by

$$F = K_2 (B_{22} - B^0_{22})^2 \qquad (40)$$

From equation (26), one can write

$$F = \gamma'^2 C^2_{1111}(B_{22} - B^0_{22})^2 + C^2_{2222}(B_{22} - B^0_{22})^2 \qquad (40)$$

Through equations (39) and (40), one finds

$$C^2_{2222} + \gamma^{*2} C^2_{1111} = K_2 \quad (41)$$

Through equations (38), (41), one can solve the coefficients C^2_{1111}, C^2_{2222} as functions of the material's constants K_1, K_2, γ^{*}.

Pure Twist Test

In such case, if the torsion constant is given by J, the strain energy in unit area is given by

$$F = J(B_{12} - B^0_{12})^2 \quad (42)$$

From equation (26), one can write

$$F = 2C^2_{1212}(B_{12} - B^0_{12})^2 \quad (43)$$

Therefore, one finds

$$C^2_{1212} = J \quad (44)$$

If the parametric material coordinates u_1 , u_2 are chosen arbitrary, the strain energy density must be expressed as a function of the measured material constants and the equivalent strain which is expressed by u_1 , u_2 firstly. Then the same methods as above can be used to determine the weighting functions.

SOLUTION SCHEME

Equation (22) is integrated in time steps of Δt. By substituting the following approximations

$$\frac{d^2\underline{\mathbf{x}}}{dt^2} \approx (\underline{\mathbf{x}}_{t+\Delta t} - 2\underline{\mathbf{x}}_t + \underline{\mathbf{x}}_{t-\Delta t}) / \Delta t^2$$
$$\frac{d\underline{\mathbf{x}}}{dt} \approx (\underline{\mathbf{x}}_{t+\Delta t} - \underline{\mathbf{x}}_{t-\Delta t}) / 2\Delta t \quad (45)$$

into equation (22), one obtains

$$\mathbf{A}_t \underline{\mathbf{x}}_{t+\Delta t} = \underline{\mathbf{g}}_t \quad (46)$$

where the MN×MN matrix $\mathbf{A}_t$ is as follows

$$\mathbf{A}_t(\underline{\mathbf{x}}_t) = \mathbf{K}(\underline{\mathbf{x}}_t) + \left(\frac{1}{\Delta t^2}\mathbf{M} + \frac{1}{2\Delta t}\mathbf{C}\right) \quad (47)$$

and the effective force vector can be written as

$$\underline{\mathbf{g}}_t = \underline{\mathbf{f}}_t + \left(\frac{1}{\Delta t^2}\mathbf{M} + \frac{1}{2\Delta t}\mathbf{C}\right)x_t + \left(\frac{1}{\Delta t}\mathbf{M} - \frac{1}{2}\mathbf{C}\right)\frac{\underline{\mathbf{x}}_t - \underline{\mathbf{x}}_{t-\Delta t}}{\Delta t} \quad (48)$$

Applying the above semi-implicit procedure, the dynamic solution can be obtained for given initial conditions $\underline{\mathbf{x}}_0$ and $\dot{\underline{\mathbf{x}}}_0$ at t=0. During each time step, a system of sparse linear algebraic equations for the instantaneous configuration $\underline{\mathbf{x}}_{t+\Delta t}$ is derived from the preceding solution $\underline{\mathbf{x}}_t$.

Construction of the Stiffness Matrix With The Quadratic Interpolation Method

Suppose that the node **x**[m,n] and its neighboring nodes constitute a quadratic surface $\mathbf{q}[m,n](t_1, t_2)$, where $-1 \le t_1 \le 1$, $-1 \le t_2 \le 1$, and

$$\begin{aligned}
&\mathbf{q}[m,n](-1,-1) = x[m-1,n-1],\\
&\mathbf{q}[m,n](-1,0) = x[m-1,n],\\
&\mathbf{q}[m,n](-1,1) = x[m-1,n+1],\\
&\mathbf{q}[m,n](0,-1) = x[m,n-1],\\
&\mathbf{q}[m,n](0,0) = x[m,n],\\
&\mathbf{q}[m,n](0,1) = x[m,n+1],\\
&\mathbf{q}[m,n](1,-1) = x[m+1,n],\\
&\mathbf{q}[m,n](1,0) = x[m+1,n],\\
&\mathbf{q}[m,n](1,1) = x[m+1,n+1].
\end{aligned} \quad (49)$$

$\frac{\partial \mathbf{q}[m,n]}{\partial t_1}$, $\frac{\partial \mathbf{q}[m,n]}{\partial t_2}$, $\frac{\partial \mathbf{q}^2[m,n]}{\partial t_1 \partial t_2}$, $\frac{\partial \mathbf{q}^2[m,n]}{\partial t_1^2}$ and $\frac{\partial \mathbf{q}^2[m,n]}{\partial t_2^2}$ can be determined from the interpolated surface and are used for the calculation of the first and second order derivatives of **x**[m,n].

The local quadratic surface is given as follow,

$$\mathbf{q}[m,n](t_1,t_2) = \begin{bmatrix} e(t_1) & f(t_1) & g(t_1) \end{bmatrix} [\mathbf{X}_{mn}]_{3\times3} \begin{bmatrix} e(t_2) \\ f(t_2) \\ g(t_2) \end{bmatrix} \quad (50)$$

where

$$[\mathbf{X}_{mn}]_{3\times3} = \begin{bmatrix} \mathbf{x}[m-1,n-1] & \mathbf{x}[m-1,n] & \mathbf{x}[m-1,n+1] \\ \mathbf{x}[m,n-1] & \mathbf{x}[m,n] & \mathbf{x}[m,n+1] \\ \mathbf{x}[m+1,n-1] & \mathbf{x}[m+1,n] & \mathbf{x}[m+1,n+1] \end{bmatrix},$$

and blending functions are

$$e(s) = \frac{1}{2}s^2 - \frac{1}{2}s$$
$$f(s) = -s^2 + 1$$
$$g(s) = \frac{1}{2}s^2 + \frac{1}{2}s$$

and the local parameters t_1 and t_2 are

$$t_1 = (u_1 - m) / h_1$$
$$t_2 = (u_2 - n) / h_2$$

Let $T_1\mathbf{x}[m,n]$, $T_2\mathbf{x}[m,n]$, $T_{12}\mathbf{x}[m,n]$, $T_{11}\mathbf{x}[m,n]$ and $T_{22}\mathbf{x}[m,n]$ represent the first and second order derivatives of **x**[m,n]. By using eqn.(50), we have

$$T_1\mathbf{x}[m,n] = \frac{\partial \mathbf{q}[m,n](0,0)}{\partial t_1} = \frac{\mathbf{x}[m+1,n]-\mathbf{x}[m-1,n]}{2h_1} \quad (51)$$

$$T_2\mathbf{x}[m,n] = \frac{\partial \mathbf{q}[m,n](0,0)}{\partial t_2} = \frac{\mathbf{x}[m,n+1]-\mathbf{x}[m,n-1]}{2h_2} \quad (52)$$

$$T_{11}\mathbf{x}[m,n] = \frac{\partial^2 \mathbf{q}[m,n](0,0)}{\partial t_1^2} = \frac{\mathbf{x}[m+1,n]-2\mathbf{x}[m,n]+\mathbf{x}[m-1,n]}{h_1^2} \quad (53)$$

$$T_{22}\mathbf{x}[m,n] = \frac{\partial^2 \mathbf{q}[m,n](0,0)}{\partial t_2^2} = \frac{\mathbf{x}[m,n+1]-2\mathbf{x}[m,n]+\mathbf{x}[m,n-1]}{h_2^2} \quad (54)$$

Construction of the Deformable Surface with C1 Connection of Coons Patches

Using the grid functions x[m,n], $T_1\mathbf{x}[m,n]$, $T_2\mathbf{x}[m,n]$, and $T_{12}\mathbf{x}[m,n]$, the corresponding bi-cubic Coons patch can be constructed as follow

$$\mathbf{x}_{mn}(t_1,t_2) = [f_0(t_1) \quad f_1(t_1) \quad g_1(t_1) \quad g_2(t_1)][\mathbf{X}_{mn}]_{4\times4}\begin{bmatrix} f_0(t_2) \\ f_1(t_2) \\ g_0(t_2) \\ g_1(t_2) \end{bmatrix} \quad (55)$$

where

$$[\mathbf{X}_{mn}]_{4\times4} = \begin{bmatrix} \mathbf{x}[m,n] & \mathbf{x}[m,n+1] & T_2\mathbf{x}[m,n] & T_2\mathbf{x}[m,n+1] \\ \mathbf{x}[m+1,n] & \mathbf{x}[m+1,n+1] & T_2\mathbf{x}[m+1,n] & T_2\mathbf{x}[m+1,n+1] \\ T_1\mathbf{x}[m,n] & T_1\mathbf{x}[m,n+1] & T_{12}\mathbf{x}[m,n] & T_{12}\mathbf{x}[m,n+1] \\ T_1\mathbf{x}[m+1,n] & T_1\mathbf{x}[m+1.n+1] & T_{12}\mathbf{x}[m+1,n] & T_{12}\mathbf{x}[m+1,n+1] \end{bmatrix}$$

and the Hermetic base functions are

$$f_0(t) = 2t^3 - 3t^2 + 1$$
$$f_1(t) = -2t^3 + 3t^2$$
$$g_0(t) = t^3 - 2t^2 + t$$
$$g_1(t) = t^3 - t^2$$

So, the whole deformable surface can be constructed by multi-patches that are connected together with C1

$$\mathbf{x}(u_1,u_2) = \{\mathbf{x}_{mn}(t_1,t_2)\} \quad (56)$$

where $1 \le m \le M-1$, $1 \le n \le N-1$.

SIMULATION EXAMPLES

The first test is an example of a square piece of cloth dropping on a ball under gravity. To prevent the cloth penetrating into the obstacle and obtain good simulation results, small time steps were used in the numerical analysis. In this example, the non-woven cotton cloth was used, whose material constants were measured using the Kawabata's Evaluation System: tensile modulus E_{u1}=21300 N/m, E_{u2}=4300 N/m; bending rigidity B_{u1}= 175 μNm/m B_{u2}= 45 μNm/m; shear modulus G=54 N/m; mass density μ[i,j]=74.5 g/m^2; and damping factor γ[i,j]=10.0g/ms. In the simulation, a 21×21 grid was used and the time step was Δt=0.01s. The results of simulation by using Terzopoulos' model and the thin flexible surface model developed in this paper were shown in Fig.1 and Fig.2 for time step at 0, 100Δt, 200Δt and 300Δt, respectively. The CPU time of the computation is about 6 minutes on a Pentium PC 200. It is obvious that the present formulation gives more realistic results.

The second example is a multi-fold skirt worn on a mannequin. In this example, the geometric model of the mannequin consists of 22 NURBS patches satisfying G1 continuity requirements.

In this test, the skirt was divided into 28×14 grids, and the skirt was assumed to be fixed at the waist line of the mannequin and the longitudinal lines on the skirt were initially straight. Figure 1 (a) shows the initial position of the skirt. Setting the constraint at the nodes on the waist line with a spring constant k=80.0, the mannequin repulsive collision constant c=120.0 and with a time step Δt=0.01s, the results of the simulation for the time step 0, 200Δt, 400Δt and 600Δt are shown in Fig.3 (a,b,c,d) ,respectively. The total CPU time is about 40 minutes on a Pentium PC 200.

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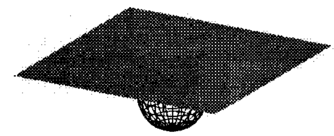

Figure 1a (t=0)

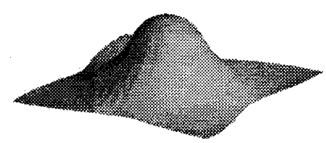

Figure 1b (t=100 Δt)

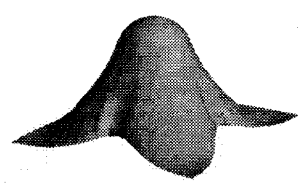

Figure 1c (t=200 Δt)

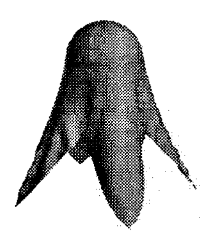

Figure 1d (t=300 Δt)

Fig.1 Cloth Falling On A Ball (Terzopoulos' Model)

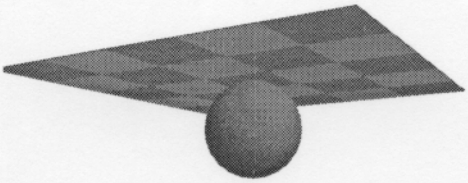

Figure 2a (t=0)

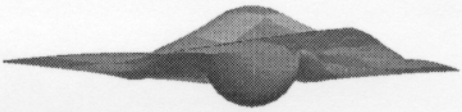

Figure 2b (t=100 Δt)

Figure 2c (t=200 Δt)

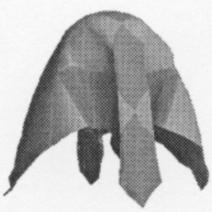

Figure 2d (t=300 Δt)

Fig.2 Cloth Falling On A Ball (Thin Flexible Surface Model)

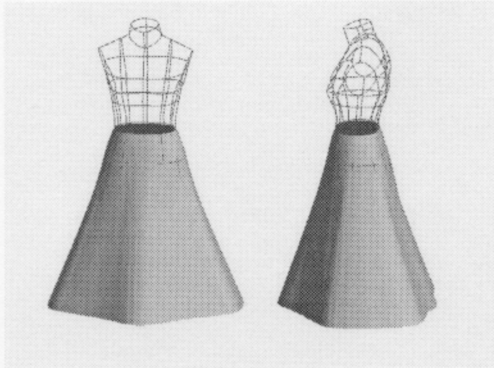

Figure 3a

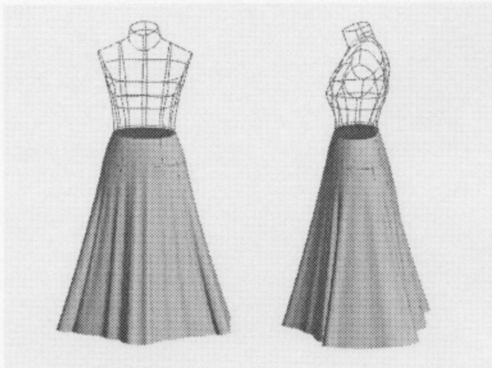

Figure 3b

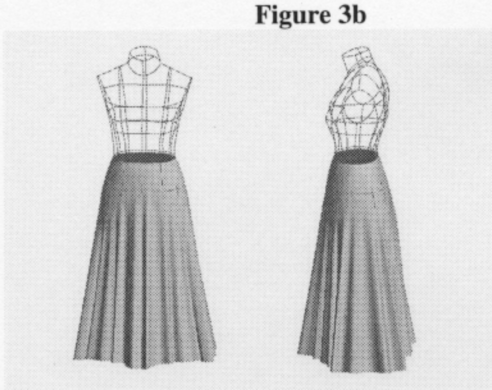

Figure 3c

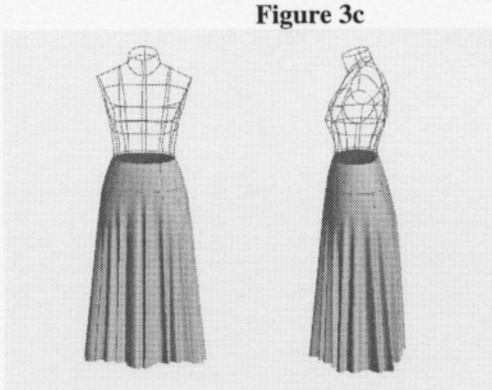

Figure 3d

Figure 3
Simulating A Multi-fold Skirt Model