ISSN 2464-4617 (print)
ISSN 2464-4625 (online)
Quantum Informatics, Computing & Technology 2025
QC-Horizon 2025
https://www.qc-horizon.eu/

Quantum Signal-Induced Heap Transform Based Realization of Multi-Qubit Operations

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ABSTRACT
The article presents a new approach to the decomposition of unitary operations by controlled rotations. A new method of quantum signal-induced heap transform based QR decomposition is described. This transform is an analogue of the discrete transform which is generated by a given signal and uses different paths of processing the data. It is shown that among such paths we can find paths that allow constructing efficient quantum circuits for implementing multi-qubit unitary gates. The case of real unitary matrices is considered. The proposed approach is described in detail in examples and quantum circuits are presented for 2-qubit operations. The general case of r-qubit unitary operations, when $r \geq 2$, is considered similarly.

Keywords
Quantum QR decomposition, quantum heap transform, quantum cosine transform.

1. INTRODUCTION
Many methods of QR-decomposition of real matrices are known. We mention the Givens rotations - [DLHL12], Gramm-Schmidt process and the method of Householder transformations [Hou58]. We stand on the method of QR decomposition. The main goal is to efficiently decompose a given operation into a set of simple gates, for example the controlled rotation gates, phase shift gates, and CNOTs.

In this work, a general method of QR decomposition is described, by using the quantum signal-induced heap transform [AM06],[AG14]. We provide a new view of the QR-decomposition of unitary matrices. The resulting codes allow us to simplify the quantum circuits for the quantum operations, including Hartley and cosine transforms.

2. THE CONCEPT OF DsiHT
The N-point DsiHT is the transform that is generated by a given signal $\boldsymbol{x}$ of length N [AM06]. The main characteristic of the DsiHT is the path, that is, the order in which it is assembled from the basic 2-point rotations of the generator elements. As example, Fig. 1 shows two diagrams of composing the 4-point DsiHT. Each unitary transform T_k, $k = 1,2,3$, is the Givens rotation, which is generally described as

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \pm\sqrt{x^2+y^2} \\ 0 \end{bmatrix}. \quad (1)$$

Here, the angle is defined by the inputs as $\vartheta = -\arctan(y/x)$, and $\vartheta = \pm\pi/2$ if $x = 0$. The path of the transform, which is shown in part (a), is the traditional path and this transform is called the DsiHT with the weak carriage-wheel (see [AG14] for detail). Two rotations are on the adjacent BPs, that is, 0 and 1, 0 and 2. The last rotation operates on BPs 0 and 3 which are not adjacent. The transform of the generator is equal to $Tx = \left(\pm\sqrt{x_0^2+x_1^2+x_2^2+x_3^2}, 0,0,0,0\right)$.

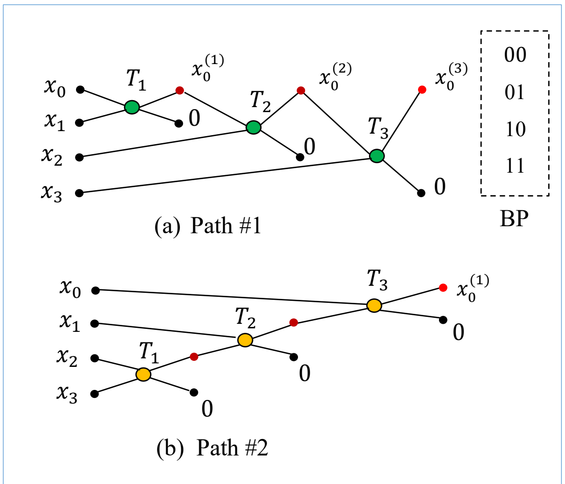

Fig. 1. Two diagrams for the 4-point DsiHTs.

The second path in part (b) also shows that one of rotations operates on the non-adjacent BPs. These BPs are 1 (01) and 2 (10). Figure 2 in part (a) shows path #3 for the 4-point DsiHT. All three rotations in this transform operate on the adjacent BPs. These BPs are 0 and 2, 1 and 3, and then 0 and 1. This path is considered good for building the circuit of the corresponding 2-qubit QsiHT, as shown in part (b).

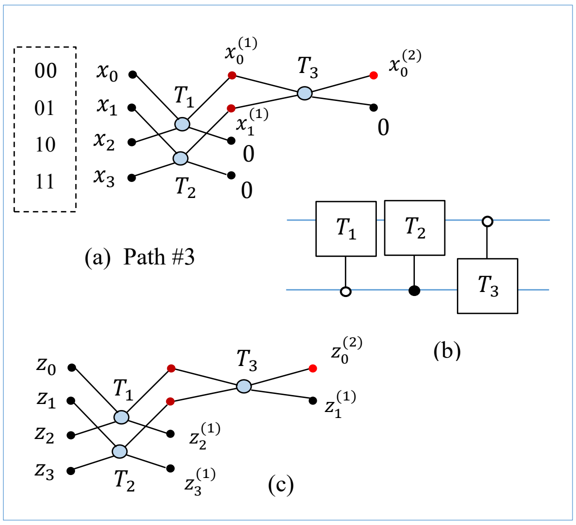

Fig. 2 The diagram and circuit for the 2-qubit QsiHT.

The above three 4-point DsiHTs, H_4, use different set of angles $A_x = \{\vartheta_1, \vartheta_2, \vartheta_3\}$ which are calculated from the same generator $\boldsymbol{x} = (x_0, x_1, x_2, x_3)$. The results of the calculations are the same (up to the sign), $H_4(\boldsymbol{x}) = \left(x_0^{(k)}, 0,0,0\right) = \left(\pm\sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}, 0,0,0\right)$, $k = 3,1,2$. The generated transform operates on an input $\boldsymbol{z} = (z_0, z_1, z_2, z_3)$ using the same path, as shown in part (c), $H_4(\boldsymbol{z}) = \left(z_0^{(2)}, z_1^{(1)}, z_2^{(1)}, z_3^{(1)}\right)$. Such effective paths exist for the N-point DsiHTs, when $N > 2$, and larger this number N, the more such paths can be found. We call them fast paths.

3. DsiHT-BASED DECOMPOSITION

In this section, we describe the QR decomposition of a square matrix A of size $2^r \times 2^r, r > 1$, by the Givens rotations. The unitary matrix A is considered with real coefficients. In the QR decomposition of the matrix A, $(2^r - 1)$ DsiHTs are used. This decomposition is illustrated below for a 4×4 unitary matrix,

$$A = \begin{bmatrix} \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ \end{bmatrix} \xrightarrow{\circ:\,DsiHT} A_1 = \begin{bmatrix} \star & 0 & 0 & 0 \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \end{bmatrix} \xrightarrow{\star:\,DsiHT}$$

$$\rightarrow A_2 = \begin{bmatrix} \star & 0 & 0 & 0 \\ 0 & \diamond & 0 & 0 \\ 0 & 0 & \diamond & \diamond \\ 0 & 0 & \diamond & \diamond \end{bmatrix} \xrightarrow{\diamond:\,DsiHT} R = \begin{bmatrix} \star & 0 & 0 & 0 \\ 0 & \diamond & 0 & 0 \\ 0 & 0 & \ast & 0 \\ 0 & 0 & 0 & \circledast \end{bmatrix}.$$

The first DsiHT, H_{0-3}, is generated by the first column of the matrix A and then transforms each of its columns. Six zero coefficients will be obtained in the new matrix A_1, as shown above. The second 3-point DsiHT, H_{1-3}, is generated by the three components of the second column of the matrix A_1. This transform is applied to the 3×3 sub-matrix and another four zero coefficients will be obtained in the new matrix A_2. The last 2-points DsiHT, H_{2-3}, is generated by the last two coefficients of the third column of A_2 and is applied to its 2×2 sub-matrix. The matrix diagonalization is complete. The matrix R is diagonal with the coefficients ± 1.

4. TWO-QUBIT OPERATIONS

Triangularization of the square matrix 4×4 in the QR decomposition by three DsiHTs can be written as

$$T: A \rightarrow R = H_{2-3} H_{1-3} H_{0-3} A. \qquad (3)$$

Here, the matrix R is a diagonal matrix with the coefficients ± 1 on the diagonal. The diagram of realization of the matrix A is given in Fig. 3.

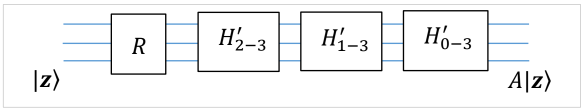

Fig. 3 Block-diagram for a 3-qubit unitary operation.

The operation A can be calculated by the inverse transform as $A = T^{-1}(R) = H'_{0-3} H'_{1-3} H'_{2-3} R$.

The case $N = 4$ (Transform H_{0-3}) The block-diagram and the quantum circuit for the 2-qubit QsiHT are given in Fig. 4.

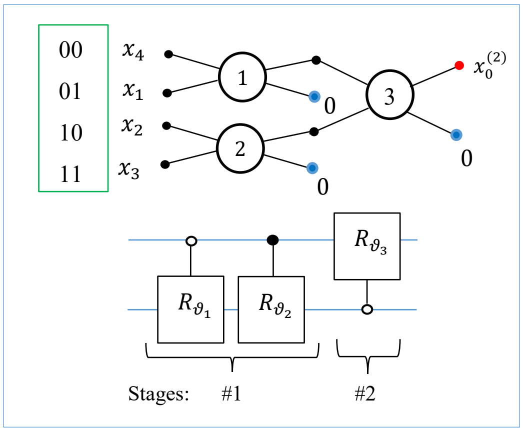

Fig. 4 The block-diagram and circuit of the 2-qubit QsiHT, H_{0-3}, on bit-planes 0-3.

The case $N = 3$ (Transform H_{1-3}) The input is the 2-qubit superposition in the form of $\boldsymbol{x}_1 = (a, x_1, x_2, x_3)$ and only the last three amplitudes are transformed (see Fig. 5). The rotation on bit-planes 2 (10) and 3 (11) is defined as $R_{\vartheta_1}: (x_2, x_3) \rightarrow (0, \pm\sqrt{x_2^2 + x_3^2})$, with $\vartheta_1 = \operatorname{atan}(x_2/x_3)$.

In the matrix form, the first circuit is described as

$$H_{1-3} = \begin{bmatrix} 1 & & & \\ & \cos\vartheta_2 & & -\sin\vartheta_2 \\ & & 1 & \\ & \sin\vartheta_2 & & \cos\vartheta_2 \end{bmatrix} (I_2 \oplus R_{\vartheta_1}).$$

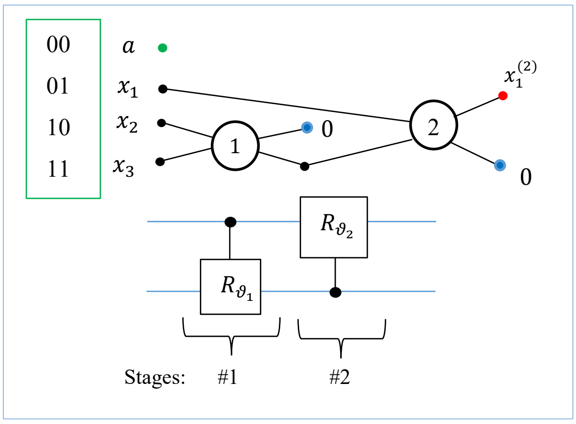

Fig. 5 The block-diagram and the circuit for the 2-qubit QsiHT, H_{1-3}, on bit-planes 1-3.

The case $N = 2$ (Transform H_{2-3}) The input is the 3-qubit superposition in the form of $\boldsymbol{x}_1 = (a, b, x_2, x_3)$ and the transform process only the last two amplitudes. The block-diagram of this 2-qubit QsiHT and the corresponding gate are shown in Fig. 7.

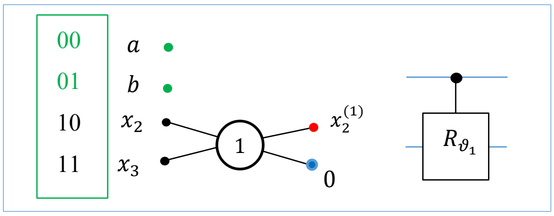

Fig. 6 The block-diagram and the circuit element for the 2-qubit QsiHT, H_{2-3}, on bit-planes 2 and 3.

All these three QsiHTs can be used in the QR decomposition of the 4× 4-matrix, $T(A) = R$. The matrix R is diagonal and the number of controlled rotation gates is equal to $\mu(r) = 3 + 2 + 1 = 6$. The circuits for realization a 2-qubit unitary operation A is given in Fig. 7. Here, $R'_{\vartheta_1} = R_{-\vartheta_k}$, for $k = 1,2$.

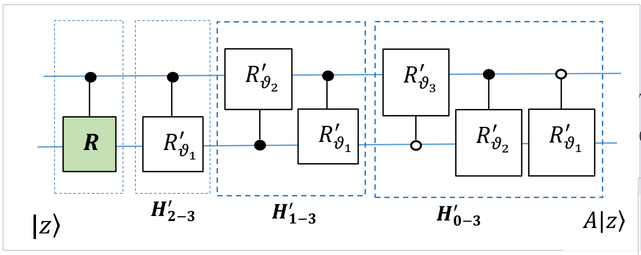

Fig. 7 The quantum circuits for 2-qubit operator.

This circuit is universal. Given a diagonal matrix R and any six angles in Table 1 can be used to generate a 2-qubit operation by the above circuit.

	ϑ_1	ϑ_2	ϑ_3
H_{0-3}	*	*	*
H_{1-3}	*	*	
H_{2-3}	*		

Table 1. 2-qubit operation angles.

If some angles are zero or 90, 180, and 270 degrees, the quantum circuit can be simplified by reducing the number of rotation gates. For comparison, the method described in [FD19] uses $4^2 = 16$ gates.

5. QUANTUM HARTLEY AND COSINE TRANSFORMS

In this section, we apply the described method for computing the quantum Hartley transform (QHyT) and quantum cosine transform (QCT) of type II.

Example 1 (The 2-qubit QCT-II) The N-point DCT-II, X_p, of a signal x_n, is calculated by [RB86]

$$X_p = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} x_n \cos\left(\frac{\pi}{N}(n + 0.5)p\right),$$
$$p = 0: (N - 1),$$

and $X_0 = X_0/\sqrt{2}$. In the $N = 4$ case, the matrix of the transform can be written as

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1.3066 & 0.5412 & -0.5412 & -1.3066 \\ 1 & -1 & -1 & 1 \\ 0.5412 & -1.3066 & 1.3066 & -0.5412 \end{bmatrix}$$

$\det A = 1$. For example, if the input vector is $z = (1, -3, 2, 3)'/\sqrt{23}$, then the DCT of this vector is equal to $y = Az$=(0.3128, -0.5546, 0.5213, 0.5682)$'$. The QR decomposition by the DsiHTs with the fast path is accomplished by the rotations with the angles given in Table 2. The matrix R in this decompositon is $R = \text{diag}\{1,1,1,1\} = I_4$, i.e., the identical matrix.

	ϑ_1	ϑ_2	ϑ_3
H_{0-3}	$-52.5708°$	$-28.4221°$	$-34.6476°$
H_{1-3}	$249.7543°$	$256.5505°$	
H_{2-3}	$-35.2644°$		

Table 2. Angles of rotations in the circuit of Fig. 7.

The circuit for the 2-qubit QCT-II includes six controlled rotation gates and is shown in Fig. 8.

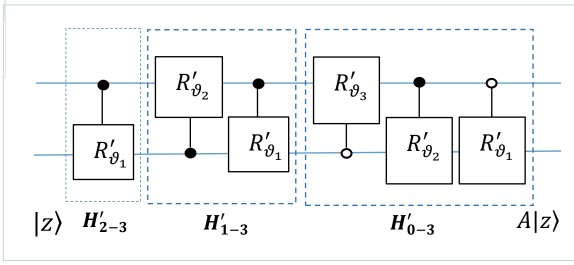

Fig. 8 The quantum circuits for 2-qubit QCT-II.

Example 2 (2-qubit quantum Hartley transform)
The matrix of the N-point transform discrete Hartley transform (DHyT) is equal to

$$A = \frac{1}{\sqrt{N}}\left[\cos\left(\frac{\pi}{N}np\right) + \sin\left(\frac{\pi}{N}np\right)\right]_{n,p=0:(N-1)}.$$

The 4-point transform has the unitary matrix

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \det A = -1. \quad (12)$$

The QR decomposition of this matrix by the DsiHTs with fast path is accomplished by the rotations with angles given in Table 3. The diagonal matrix is equal to $R = \text{diag}\{1,1,1,-1\} = I_2 \oplus Z$.

	ϑ_1	ϑ_2	ϑ_3
H_{0-3}	$-45°$	$-45°$	$-45°$
H_{1-3}	$270°\ (-90°)$	$-90°$	
H_{2-3}	$225°$		

Table 3. Angles of the rotation for 2-qubit QHyT.

Because of angles of 45°, the circuits of the 2-qubit QHyT can be simplified as shown in Fig. 9. It includes four controlled rotation gates and one local rotation.

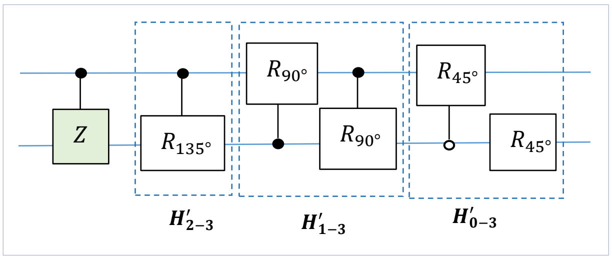

Fig. 13 The quantum circuits of the 2-qubit QHyT.

A. Results of Qiskit Simulation

To validate the proposed quantum signal-induced heap transform (QsiHT)-based QR decomposition, the quantum algorithms were implemented and simulated using IBM's Qiskit framework. In the following tables, the QCT-II and QHyT simulation results are recorded. For each simulation, the circuit was executed for different numbers of shots (1,000, 10,000, and 100,000) to analyze the convergence of results compared to the theoretical expectation. The measurement results were analyzed using the Mean Relative Squared Error (MRSE) metric.

	Ideal	1,000 shots	10,000 shots	100,000 shots
$\|00\rangle$	0.3127	0.2932	0.3122	0.3116
$\|01\rangle$	0.5545	0.5718	0.5494	0.5560
$\|10\rangle$	0.5212	0.5310	0.5189	0.5201
$\|11\rangle$	0.5682	0.5523	0.5757	0.5684
MRSE	0.0000	8.0230e-03	2.3438e-03	5.7055e-04

Table 4: Qiskit results of the 2-qubit QCT-II shown in Example 1.

	Ideal	1,000 shots	10,000 shots	100,000 shots
$\|00\rangle$	0.3127	0.3224	0.3153	0.3120
$\|01\rangle$	0.7298	0.7197	0.7328	0.7310
$\|10\rangle$	0.3127	0.3127	0.3173	0.3116
$\|11\rangle$	0.5212	0.5282	0.5126	0.5211
MRSE	0.0000	3.9322e-03	2.6370e-03	5.3517e-04

Table 5: Qiskit results 2-qubit of the quantum Hartley transform shown in Example 2 with $z = (1,-3,2,3)'/\sqrt{23}$ as input.

6. CONCLUSION

In this work, we describe the method of QsiHT-based QR decomposition for unitary operations. The quantum circuits for implementing unitary operations on two-qubit superpositions are presented. The corresponding quantum circuit for unitary operations on 2 qubits are presented. Examples with the Hartley and cosine transformations are also considered. The presented method can also be used to construct quantum circuits for m-qubit operations, when $m \geq 2$, since the fast paths with splitting for m-qubit DsiHT can be also found [AG25]. For 3-qubit operations, our quantum circuit uses a maximum of 28 rotations.

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