

Extending Bloch Sphere model to a 2-qubits system

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ABSTRACT

In this paper, we present a novel representation of two-qubit systems using three Bloch spheres. We explicitly construct a bijection between the Hilbert space of state vectors and a triple-sphere framework, where the first two Bloch spheres encode the individual qubit states, and the third sphere captures the entanglement between them. This geometric interpretation provides intuitive insights into the structure of composite quantum systems and lays the groundwork for further generalization to n -qubit systems. This work is the first step in a broader line of research aimed at developing a Bloch-sphere-based model to describe and analyze the behavior of multi-qubit systems, with a special focus on the study of entanglement.

Keywords

Quantum computing - Quantum Information - Bloch Sphere - Hilbert Space

1 INTRODUCTION

The idea of a quantum computer was first proposed by Prof. Richard Feynman [Fey82] in 1982, who pointed out that accurately and efficiently simulating quantum mechanical systems would be almost impossible on a classical computer, but not on a new kind of machine, a computer itself built of quantum mechanical elements which obey quantum mechanical laws.

There exist several models of computation, among which the circuit model is the most widely used for quantum computing [E.89, PJ08]. In order to formulate a Linear Algebra for this model, scalars, vectors and matrices should be defined. For the first, $\mathbb{C}$ is the answer. Let's introduce the other two in detail.

Classical information is based on the notion of bit, a base-2 number that takes either the value 1 or the value 0, meanwhile quantum information is based on qubits, which are represented in a similar way since they are

also base-2 numbers so showing the same values 1 or 0 when measured and therefore collapsed to a classical state. Nevertheless, qubits may also be in a sort of superposition state which conforms to a two-dimensional vector in the complex space $\mathbb{C}^2$, with orthonormal basis vectors $|0\rangle$ and $|1\rangle$. This conforms a very appropriate way to mathematically represent the state of a qubit at any given time within a quantum system. Equation 1 shows the state of the qubit $|\psi\rangle$ represented as a linear combination of those basis vectors.

$$|\psi\rangle = a|0\rangle + b|1\rangle \quad a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \quad (1)$$

Where a is the complex scalar amplitude of measuring the basis state $|0\rangle$, and b is the same for measuring the value $|1\rangle$. Amplitudes may be thought of as quantum probabilities that measure the chance with which a given quantum state will be observed when the superposition is made to collapse.

A widely used model in quantum information processing is the *Bloch sphere representation*, where a simple qubit state $|\psi\rangle$ is represented by a point located in the surface of a standard unitary sphere, whose coordinates are expectation values of physically interesting operators for the given quantum state. Operations on individual qubits commonly used in quantum information processing can also be represented in the Bloch sphere. Within the Bloch sphere, the north and the south pole

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are defined as the states of the orthonormal base $|0\rangle$ and $|1\rangle$, respectively. Any unit operation, which leads from an initial state to the final state of the single qubit, is equivalent to a composition of rotations on the axes of the Bloch sphere.

Due to equivalent representations of states via the Bloch sphere of figure 1, any state of a single qubit can be written as:

$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\varphi} \sin \frac{\theta}{2} |1\rangle \quad \theta \in [0, \pi], \varphi \in [0, 2\pi] \quad (2)$$

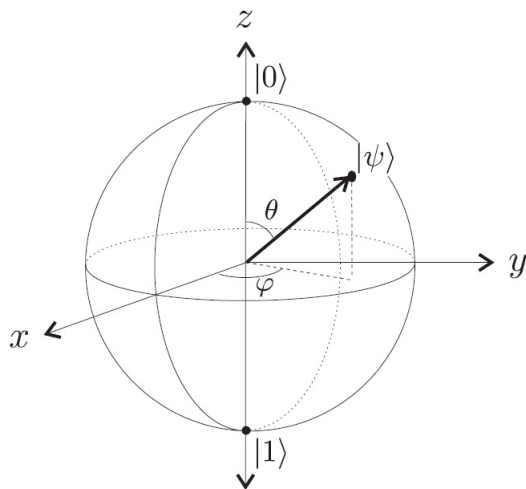


Figure 1: State of a qubit on the Bloch sphere

The Bloch sphere can be used to easily visualize the effect of applying quantum gates, or, the temporal evolution of the state of a two-level system described by a Hamiltonian, as in studying the pulses used in nuclear magnetic resonance. In both cases, the effect of applying a 2×2 unitary matrix (which can always be decomposed as a product of rotation operators) must be studied.

A rotation operator is defined by an axis and an angle of rotation. The action of a rotation operator on the quantum state translates, the point associated with the state on the Bloch sphere, into the point reached once rotated the given angle over the axis of rotation.

It is clear that some geometric pictures help us greatly in understanding some quantum information or quantum algorithms problems, such as the representation of a unit vector of the Bloch sphere. This representation is smart and powerful for a single qubit. Nevertheless, there is no commonly accepted representation for a quantum system for two or more qubits so far, even when we can find some papers reported in the literature on the subject.

Thus, in this paper we propose a model extension of the Bloch Sphere for a system conformed by more than one qubit, where for the case of 2 qubits, exactly 3 spheres are needed, so showing the way in which the spheres are interacting. Despite the length of some formulae in it, the proposed model is quite simple and, with no doubt, means a clear advantage for describing, analyzing and processing tasks.

The paper is organized as follows. First, we introduce some fundamental concepts regarding the Bloch sphere representation for a single qubit. Next, we extend the discussion to the two-qubit system and explain the key ideas behind this representation. Following this, we present and prove a theorem that establishes a bijection between the vector representation and the three Bloch spheres representation. Finally, we provide several examples to illustrate the potential of this alternative representation.

2 BACKGROUND

We can find in the scientific literature not many papers studying different models for representing general sets of qubits, in particular, the basic case of two qubits. Among them, we can cite [MD01] as the preliminary work, where a generalization of the standard Bloch sphere representation for two qubits is presented in the framework of Hopf fibrations of high-dimensional spheres by lower-dimensional ones. There, they need a 7-dimensional sphere for the model. In [HD04] it is shown that the states of a system of two qubits can be represented in a 6-dimensional geometric algebra quite similar to the Bloch Sphere. In [Wha16] it is reported that any pure two-qubit state can be represented by six real angles, with a natural parameterization induced by the bipartite structure. Up to a certain point this is a result close to ours, but there are some differences that we remark on in the following. And finally, we can cite [Wie20], where a model of three Bloch Spheres is presented as a model for a 2 qubits system, by means of Hopf fibrations, as the first work cited, besides the fact that one of the spheres is not unitary, thus they need an additional dimension to represent the radius of the sphere, so getting in the end 7 dimensions. Wang [Wan18] applied a geometric algebra to analyze the space of a multi qubits system, particularly two and three, so using a single angle to represent the entanglement in terms of the Von Newman entropy. Finally, in [DGB22] a model of a two qubit system is presented, where two spheres and a 3×3 correlation matrix are used.

After the study of the related work, we can see that there are proposed models to represent 2 qubits by using 6 or 7 real dimensions.

2.1 The model for a single qubit.

According to eqn. 1, two complex values are required, i. e. four real values corresponding to the real part and the imaginary one of each. Since the sum of the squares of the modules of these complex numbers must be one, one of their associated four real values could be computed from the rest, so that, we will only need three real dimensions to describe the state of a qubit.

However, the representation of eqn. 2 only uses two real values to represent both angles, θ and φ . This is due (see Chapter 1 of [NC11]) to the fact that from our perspective, the state of a qubit does not change when it is multiplied by a *global phase* γ since the resulting one is indistinguishable from the former. Therefore, eqn. 1 translated into polar coordinates remains this way

$$|\psi\rangle = r_1 e^{i\gamma} |0\rangle + r_2 e^{i(\gamma+\varphi)} |1\rangle \quad (3)$$

Where $r_1, r_2 \in [0, 1] \wedge r_1^2 + r_2^2 = 1$, $\arg(a) = \gamma$ and $\arg(b) = \gamma + \varphi$.

According to previous paragraph, if we multiply the qubit $|\psi\rangle$ by a constant multiplier of the form $e^{-i\gamma}$, the observed value does not change

$$|\psi\rangle = e^{-i\gamma} |\psi\rangle = r_1 |0\rangle + r_2 e^{i\varphi} |1\rangle \quad (4)$$

Finally, if we rename $r_1 = \cos \frac{\theta}{2}$, and $r_2 = \sin \frac{\theta}{2}$, we obtain eqn. 2, so showing that two real dimensions is enough to represent a single qubit.

2.2 The model for two separated qubits

Let's proceed in similar terms. In this case, the base of the Hilbert space to represent the state ψ of a system of two qubits is $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \quad (5)$$

Where $a, b, c, d \in \mathbb{C}$, $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$.

We start from 8 real dimensions to represent the state of a system of two qubits. As before, one of them is depending on the others from the sum of the square of the modulus being one, therefore we just need 7 dimensions.

In addition, as the global phase factor is still held, the first non-zero element of the vector can be assumed to be a real number with no imaginary component (after multiplying the entire vector by the inverse of its phase). Therefore, we will only need 6 real dimensions.

In the same way, given two qubits represented by their Bloch spheres (see eq. 2) when we assume the separable system formed by them both, it is clear that the quantum state of the system can be obtained from the

tensor product of the corresponding quantum states of the components as it can be seen below

$$\begin{aligned} |Q_1\rangle &= \cos \frac{\theta_1}{2} |0\rangle + e^{i\varphi_1} \sin \frac{\theta_1}{2} |1\rangle \\ |Q_2\rangle &= \cos \frac{\theta_2}{2} |0\rangle + e^{i\varphi_2} \sin \frac{\theta_2}{2} |1\rangle \\ |Q_1\rangle \otimes |Q_2\rangle &= \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} |00\rangle + \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i\varphi_2} |01\rangle + \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{i\varphi_1} |10\rangle + \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i(\varphi_1+\varphi_2)} |11\rangle \end{aligned} \quad (6)$$

In this case, we do not have any kind of entanglement, so only 4 real dimensions are required to represent the system.

Now we want to extend this result to the general case which may include entanglement, in next section.

3 MODELLING THE 2 QUBITS SYSTEM

From the literature referred to in the previous section, we can observe that the models for a 2 qubit system are using either 6 or 7 real dimensions in order to describe themselves. We have stated that 6 would be enough for the reasons previously exposed. In this section we will see that these 6 dimensions can be captured by means of 3 Bloch spheres, each composed by two angles, respectively (θ_1, φ_1) and (θ_2, φ_2) for encoding each of the 2 qubits, and the last one (θ_3, φ_3) devoted to modelling the entanglement between them. The idea we follow is quite simple and visual.

In [NC11] (section 1.3.1) it is shown that an arbitrary single qubit unitary gate can be decomposed in the following rotation gates along Z and Y axes.

$$R_z(\varphi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{bmatrix} \quad R_y(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad (7)$$

Furthermore, we start from eq. 6, which corresponds to a quantum system of 2 separable qubits, i.e. entanglement-free, for which just 2 Bloch spheres would be enough for modeling.

In order to start considering entanglement between the qubits, we split the options in this way:

Entanglement in the phase component Let's analyze the phases of the 4 components of the vector representing the quantum state.

- $|00\rangle$ component lacks of a phase because it is the tensor product of the $|0\rangle$ components of each of the qubits
- Amplitude of $|01\rangle$ has a phase, which is the translation of the phase of qubit Q_2

- Amplitude of $|10\rangle$ has a phase, which is the translation of the phase of qubit Q_1
- Amplitude of $|11\rangle$ has a phase equal to the sum of the two previous phases

It is immediate that when dealing with a separable state, the phase of amplitude for $|11\rangle$ can be computed from those of $|01\rangle$ and $|10\rangle$.

Nevertheless, considering an arbitrary vector (a, b, c, d) belonging from $\mathbb{C}^4$ with no phase in the first component (a), the phase of the fourth component can be independent from the rest, therefore we need to introduce the phase for the amplitude of vector $|11\rangle$, as an additional component namely, from now on, the entanglement phase.

For dealing with this we adopt a notation inspired in [BBC⁺95] for a controlled binary gate defined as follows.

Definition 1 Let $|Q_1 \dots Q_n\rangle$ be a system of n qubits, let U be a binary gate acting over a single qubit, let $i \in \{2 \dots n\}$ be the index inside the register of the qubit over which U acts, and let $J \subset \{1 \dots i-1\}$ be a set of qubits. We denote the controlled gate acting on qubit i controlled by qubits from set J , as $\bigwedge_i^J U$.

□

In the particular case in which $J = \emptyset$, for a system of n qubits, we have a not controlled gate, thus $\bigwedge_i^\emptyset U = I^{\otimes i-1} \otimes U \otimes I^{\otimes n-i-1}$.

According to this notation, the introduction of the above-mentioned new phase will be denoted by a new phase angle φ_3 only acting over the $|11\rangle$ amplitude, in a way similar to the following quantum rotation gate, now controlled on qubit 1.

Thus, $\bigwedge_2^1 R_z(\varphi_3)$ defined in this way

$$\bigwedge_2^1 R_z(\varphi_3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\varphi_3} \end{bmatrix} \quad (8)$$

which defines the behaviour of the phase component of the third Bloch sphere defining the system of two qubits.

Now, we will proceed to define the behaviour of the remainder component of the third sphere, the angle θ_3 , on the real part of the amplitudes $|10\rangle$ and $|11\rangle$.

To do that, we can see from the modulus of the amplitudes of those basis vectors, in a general quantum state of two qubits, that its final amplitudes cannot depend only on the amplitudes of the components if they are not separable. Thus, we need a *redistribution* of the

corresponding values. This can be made in a similar way to the work of CNOT gate, which is the easier way to obtain a pair of entangled qubits, the so-called Bell states, represented in eqn. 9.

$$|\beta_{00}\rangle = \frac{\sqrt{2}}{2} |00\rangle + \frac{\sqrt{2}}{2} |11\rangle \quad (9)$$

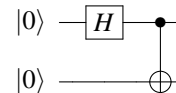


Figure 2: Bell State

As it is well known, there exists no pair of qubits Q_1, Q_2 such that the separable system formed with them being the Bell state, because from eqn. 6, we can see that no pair of θ_1 and θ_2 angles may produce Bell state as their tensorial product.

A Bell state may be obtained, as indicated in fig. 2, by two qubits initialized to $|0\rangle$, then applying them a Hadamard and a CNOT gates, producing the entanglement. In this case, the last gate interchanges amplitudes of vector basis $|10\rangle$ and $|11\rangle$.

We use the behavior of CNOT gate as inspiration, but it is not enough to reach all possibilities of redistribution among the amplitudes of $|10\rangle$ and $|11\rangle$, thus, we introduce a general gate to do this task, the controlled on qubit one version of rotation R_y which we will call $\bigwedge_2^1 R_y(\theta)$, defined as the matrix

$$\bigwedge_2^1 R_y(\theta_3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \theta_3 & -\sin \theta_3 \\ 0 & 0 & \sin \theta_3 & \cos \theta_3 \end{bmatrix} \quad (10)$$

Thus, given two qubits $|Q_1\rangle, |Q_2\rangle$ and their tensorial product $|Q_1\rangle \otimes |Q_2\rangle$ in a separable state, we can obtain all the possible combinations of modulus in the composition vector by applying the gate $\bigwedge R_y$. And, after that, the rest of the possible phases in the component of the vector by applying the gate $\bigwedge R_z$, in both cases with the corresponding angles.

Finally, we have that for a general quantum system of two qubits, if the 3 spheres for it are respectively $\{(\theta_1, \varphi_1), (\theta_2, \varphi_2), (\theta_3, \varphi_3)\}$, then the system described by eq. 5 is now

$$\begin{aligned}
 |Q_1 Q_2\rangle &= \Lambda_z^1 R_z(\varphi_3) \cdot \Lambda_z^1 R_y(\theta_3) \cdot (|Q_1\rangle \otimes |Q_2\rangle) \\
 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\varphi_3} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \theta_3 & -\sin \theta_3 \\ 0 & 0 & \sin \theta_3 & \cos \theta_3 \end{bmatrix} \cdot ((\cos \frac{\theta_1}{2} |0\rangle + e^{i\varphi_1} \sin \frac{\theta_1}{2} |1\rangle) \otimes (\cos \frac{\theta_2}{2} |0\rangle + e^{i\varphi_2} \sin \frac{\theta_2}{2} |1\rangle)) \\
 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\varphi_3} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \theta_3 & -\sin \theta_3 \\ 0 & 0 & \sin \theta_3 & \cos \theta_3 \end{bmatrix} \cdot (\cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} |00\rangle + \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i\varphi_2} |01\rangle + \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{i\varphi_1} |10\rangle + \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i(\varphi_1+\varphi_2)} |11\rangle)
 \end{aligned} \quad (11)$$

Note that, as previously commented, in [NC11] is shown that every state of a single qubit can be reached by rotations indicated by two angles (θ, φ) , and now we will extend this idea showing that every possible entanglement, also, may be described in terms of a similar *rotation* described by other two angles.

Theorem 1 Let $|Q_1 Q_2\rangle$ be a system of two qubits, represented by a vector in the Hilbert space of basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, with complex components $[a, b, c, d]$. Then, it is equivalent to represent the state by means of 6 angles, $\{(\theta_1, \theta_2, \theta_3) \in [0, \pi], (\varphi_1, \varphi_2, \varphi_3) \in [0, 2\pi]\}$. That is, by three Bloch spheres, where the three first angles represent the longitude (Z axis), and the other three the latitude (XY plane) of their corresponding sphere, as it is shown in fig. 1.

Proof: We know that a single qubit may be represented by one Bloch sphere, as it is shown in eqn. 4. Thus, in this case there exist a bijection between the coordinates in eqn. 1 and the angles in the system of the three Bloch spheres.

To prove our theorem, we proceed in three parts. In the first one, we will show the bijection between the first two amplitudes (a, b) of the vector and the angles of the Bloch spheres of the two qubits, in case there are real values with no phase.

In the second part, we will show the bijection between the real part of the third and fourth amplitudes (c, d) , and the angle θ_3 of the third sphere, in the same case considering no phase values.

Finally, in part 3, we will show the correspondence between the phase of the fourth amplitude (d) and the angle φ_3 of the third sphere.

Part 1. In this part, let us suppose amplitudes are in the real space, and we have not in consideration the phases. Thus, according to eqn. 5, we have 4 positive real values (modules of the complex values) such that

$$a^2 + b^2 + c^2 + d^2 = 1$$

In this case, as we are omitting the phase angles, we have only three angles: $\{\theta_1, \theta_2, \theta_3\}$ in the three spheres.

We also consider, in this part, that the state of two qubits has no entanglement, and therefore it consists of the tensorial product of the two qubits states $|Q_1\rangle$ and $|Q_2\rangle$, each one represented by a Bloch sphere, respectively (θ_1, φ_1) and (θ_2, φ_2) .

Then the components a and b of the vector representing the state of both qubits can be represented, by the angles θ_1 and θ_2 of the Bloch spheres. There is a bijection between both representations.

That is, given $a, b \in [0, 1]$ with $a^2 + b^2 \leq 1$, and $\theta_1, \theta_2 \in [0, \pi]$

Then, the function

$$f : [0, \pi]^2 \longrightarrow [0, 1]^2$$

defined as

$$f(\theta_1, \theta_2) = (a, b) = (\cos \theta_1/2 \cdot \cos \theta_2/2, \cos \theta_1/2 \cdot \sin \theta_2/2)$$

is bijective for all values of the domain and range, except $\theta_1 = \pi$.

To show that, we can see that the components a and b represent the coordinates of a point in the interior of the quadrant of the circle of radius 1 (See fig. 3). And then, this condition is satisfied

$$\cos^2 \theta_1/2 = a^2 + b^2$$

Whereas, in the triangle with sides $\{a, b, \cos \theta_1/2\}$ represented in fig. 3 we have that $\sin \theta_2/2$ corresponds to a and $\cos \theta_2/2$ corresponds to b .

Therefore, and as it is shown in the figure, we can establish the indicated bijection f , which corresponds to the description of the points inside the quadrant of the circle with radius 1.

Thus, with the bijection f we have that is equivalent the representation of the first half of the Hilbert space vector (a, b) , in their modulus of the complex numbers, or by means of the angles θ_1 and θ_2 , with the exception when $\theta_1 = \pi$, in which $a = b = 0$, and angle θ_2 may take any value. In this case, as all of them are equivalent, we take $\theta_2 = 0$ as the canonical representative.

Part 2. In this part, we extend the bijection f to cover all the amplitudes of the vector of the Hilbert space (a, b, c, d) , but, as in the previous part, we consider only the real part, skipping the phase, which will be dealt with in the next part.

The amplitudes c and d of the state vector representing a quantum state of qubits Q_1 and Q_2 depend uniquely

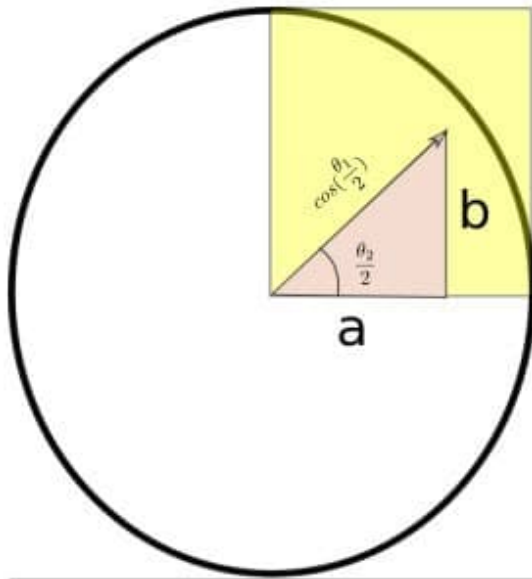


Figure 3: Correspondence between (a, b) and (θ_1, θ_2)

on a and b if the state is separable, since a separable vector, as we have in eqn. 6 has the form

$$(a, b, c, d) = (\cos \theta_1 \cos \theta_2, \cos \theta_1 \sin \theta_2, \sin \theta_1 \cos \theta_2, \sin \theta_1 \sin \theta_2) \quad (12)$$

Due to the fact that we are considering phase components $\varphi_i = 0$ in this part.

In a separable state, therefore, we have verified that with the angles θ_1 and θ_2 , which represent the modules of the Bloch sphere of both qubits, we can reach all the possible values of the components a and b of the vector. However, c and d amplitudes are predetermined by the above values. The idea, now, will be to incorporate a new sphere S_3 , with its two angles θ_3 and φ_3 , which will represent the module and its phase. For now, in this part 2, we will focus on the action of the angle θ_3 , which will result in redistributing the unique values for modulus of the amplitudes c and d , predetermined by the tensorial product of the qubits Q_1 and Q_2 , in such a way that amplitudes (c, d) can map on all possible values in the space $[-1, 1] \times [-1, 1]$, maintaining the restriction that sum of squares are equal to 1.

To do this, we consider eqn. 11, where on the column vector (a, b, c, d) , which will represent the initial state obtained through the tensor product of $|Q_1\rangle$ and $|Q_2\rangle$, is acting the quantum gate $\wedge^1 R_y(\theta_3)$ defined in eqn. 10

Thus, we need to check that the angles $(\theta_1, \theta_2, \theta_3)$, ranging each one in values in the interval $[0, \pi]$ will allow us to map the range of all the possible values of the components of the vector (a, b, c, d) , taking into account that the sum of the squares is 1, and therefore the last amplitude is predetermined by the other 3.

To do this, we have that amplitudes c and d from application of eqn. 11 are

$$\begin{aligned} c &= \sin \theta_1/2 \cdot \cos \theta_2/2 + \cos \theta_3 - \sin \theta_1/2 \cdot \sin \theta_2/2 \cdot \sin \theta_3 \\ d &= \sin \theta_1/2 \cdot \cos \theta_2/2 + \sin \theta_3 + \sin \theta_1/2 \cdot \sin \theta_2/2 \cdot \cos \theta_3 \end{aligned} \quad (13)$$

That is

$$\begin{aligned} c &= \sin \theta_1/2 \cdot (\cos \theta_2/2 + \cos \theta_3 - \sin \theta_2/2 \cdot \sin \theta_3) \\ d &= \sin \theta_1/2 \cdot (\cos \theta_2/2 + \sin \theta_3 + \sin \theta_2/2 \cdot \cos \theta_3) \end{aligned} \quad (14)$$

And, applying the definition of the trigonometric functions of the sum of angles

$$\begin{aligned} c &= \sin \theta_1/2 \cdot \cos(\theta_2/2 + \theta_3) \\ d &= \sin \theta_1/2 \cdot \sin(\theta_2/2 + \theta_3) \end{aligned} \quad (15)$$

From eqn. 15 we have that, as θ_2 and θ_3 range in interval $[0, \pi]$ the sum of $\theta_2/2 + \theta_3$ range in interval $[0, 3\pi/2]$. And then, the value of $\cos(\theta_2/2 + \theta_3)$ maps in the range from $\cos \theta_2/2$ to $\cos(\theta_2/2 + \pi)$, that is, all the possible values in the interval $[-1, 1]$. The same can be said about the value of $\sin(\theta_2/2 + \theta_3)$. We can see it in the fig. 4.

In particular, the value of amplitude c reaches its minimum, in absolute value, when $\theta_3 = \pi/2 - \theta_2/2$ in which case is 0. And the maximum (in absolute value) when $\theta_3 = \pi - \theta_2/2$, when it is $-\sin \theta_1/2$.

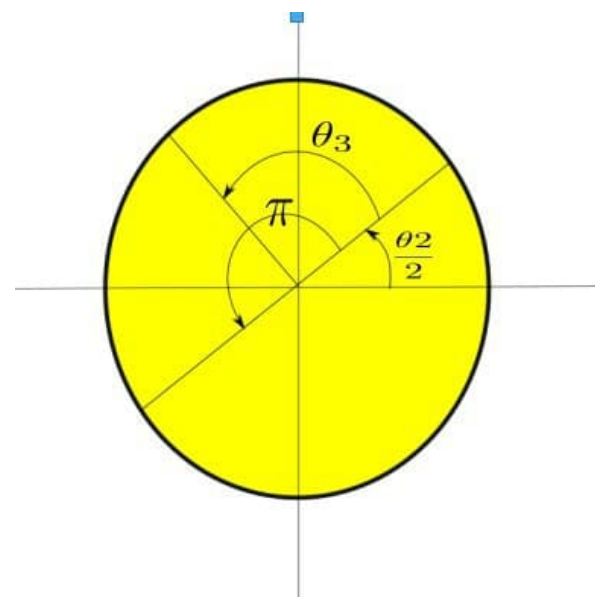


Figure 4: Range of $\theta_2/2 + \theta_3$

Then, we can extend the function f defined as

$$f : [0, \pi]^3 \longrightarrow [0, 1]^2 \times [-1, 1]^2$$

$$f(\theta_1, \theta_2, \theta_3) = (a, b, c, d) =$$

$$(\cos \theta_1/2 \cdot \cos \theta_2/2, \cos \theta_1/2 \cdot \sin \theta_2/2,$$

$$\sin \theta_1/2 \cdot \cos(\theta_2/2 + \theta_3), \sin \theta_1/2 \cdot \sin(\theta_2/2 + \theta_3),)$$
(16)

Then f is bijective for all values of the domain and range, except $\theta_1 = \pi$ and $\theta_1 = \pi/2$. In this case, there is no possibility of entanglement, and every value for θ_3 is possible; thus, as all of them are equivalent, we take $\theta_3 = 0$ as the canonical representative.

We need to note that in parts 1 and 2, we are considering the modulus of the amplitudes, and they are positive real numbers. But in this case, as we have the sum of angles $\theta_2 + \theta_3$ ranging on the interval $[0, 3\pi/2]$, then both the sine and the cosine of that sum can take negative values, and thus amplitudes c or d may be negative. However, and as we will see in the next part of the proof, this will not be a problem, because this question of the sign will be treated as a change of the phase.

Part 3 We have already verified that the the 3 angles θ_i establish a bijection for us in the modulus of the four complex amplitudes of the vector (a, b, c, d) , with the exception of the sign of a component, and taking into account that the sum of the square of the values is equal to 1. Now, to finish the proof, we need to take into account the phase angles, and assume the general case that the vector is not of real values, but complex.

First of all, we must remember from eqn. 4 that the first amplitude, namely a , is not a complex value but a positive real, and only the rest of the amplitudes, b, c, d , will have an imaginary component (phase).

The phase is acting in the Bloch sphere, as it is indicated in eqn.2. Until now, we have considered that the phase angles have zero value, and we have only taken into account the latitude angles, corresponding to the modules of the amplitudes. This last part will be relatively straightforward.

As indicated in 6, the tensorial product of two Bloch spheres gives a quantum separable state.

$$|Q_1 Q_2\rangle = \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} |00\rangle +$$

$$\cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i\varphi_2} |01\rangle +$$

$$\sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{i\varphi_1} |10\rangle +$$

$$\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i(\varphi_1 + \varphi_2)} |11\rangle$$
(17)

As we can see, the phase of the qubit Q_1 is transferred to the amplitude of the base element $|10\rangle$ of the Hilbert space, while the phase of the qubit Q_2 does so on the amplitude of the element $|01\rangle$. Finally, the phase of the base element $|11\rangle$ is predetermined by the previous ones, corresponding to the sum of both phases.

Therefore, there exists a bijection between the phases of the component qubits of a two-qubit system, and the phases of the amplitudes b and c of the vector representing the composition state. While the phase of the last component is predetermined by the previous ones.

Therefore, given two qubits and their tensorial product in a separable state, we can obtain the rest of the states by applying the gate $\wedge^1 R_z(\varphi)$, shown in eqn. 8, where φ ranges in the interval $[0, 2\pi]$ to reach all possible phases for amplitude d , in order to do it independent of the phases of the qubits components of the system.

Thus, we have that every quantum state of two qubits can be represented by 3 Bloch spheres, that is, 6 angles, in the form indicated in eqn. 11.

And finally, we have the bijection

$$f : [0, \pi]^3 \times [0, 2\pi] \longrightarrow \mathbb{C}^4$$

defined as

$$f((\theta_1, \varphi_1), (\theta_2, \varphi_2), (\theta_3, \varphi_3)) = (a, b, c, d) =$$

$$(\cos \theta_1/2 \cdot \cos \theta_2/2, \cos \theta_1/2 \cdot \sin \theta_2/2 \cdot e^{i\varphi_2},$$

$$\sin \theta_1/2 \cdot \cos(\theta_2/2 + \theta_3) \cdot e^{i\varphi_1},$$

$$\sin \theta_1/2 \cdot \sin(\theta_2/2 + \theta_3) \cdot e^{i(\varphi_1 + \varphi_2 + \varphi_3)})$$
(18)

which shows the equivalence of both representations. □

Example 1 In Fig. 2 we can see the circuit for a Bell State ($|\beta_{00}\rangle$). The system can be described by the following spheres

$$S_1 = (\pi/2, 0)$$

$$S_2 = (0, 0)$$

$$S_3 = (\pi/2, 0)$$
(19)

$$|\beta_{00}\rangle = \wedge_2^1 R_z(0) \cdot \wedge_2^1 R_y(\pi/2) \cdot \begin{bmatrix} \cos \frac{\pi}{4} \cos 0 \\ \cos \frac{\pi}{4} \sin 0 \\ \sin \frac{\pi}{4} \cos 0 \\ \sin \frac{\pi}{4} \sin 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$
(20)

□

Example 2 In Fig. 5 we can see the circuit for an entanglement in the Z axis. Again, the system can be described by three spheres, as we will indicate.

In this case, we have two qubits in a superposition state by applying a Hadamard gate on each one, and then, a

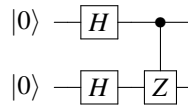


Figure 5: Z entanglement

controlled Z rotation in the amplitude of $|11\rangle$ basis component. As the Bell State, this quantum system cannot be reached by two separate qubits, but it can be represented by the following three spheres.

$$\begin{aligned} S_1 &= (\pi/2, 0) \\ S_2 &= (\pi/2, 0) \\ S_3 &= (0, \pi) \end{aligned}$$

$$|Z_{ent}\rangle = \bigwedge_2^1 R_z(\pi) \cdot \bigwedge_2^1 R_y(0) \cdot \begin{bmatrix} \cos \frac{\pi}{4} \cos \frac{\pi}{4} \\ \cos \frac{\pi}{4} \sin \frac{\pi}{4} \\ \sin \frac{\pi}{4} \cos \frac{\pi}{4} \\ \sin \frac{\pi}{4} \sin \frac{\pi}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \quad (21)$$

Corollary 1 Let $|Q\rangle = (a, b, c, d)$ be a vector in the Hilbert Space of basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ with real components, then there exists one unique set of three spheres, with phase 0 and modulus $(\theta_1, \theta_2, \theta_3)$ such that bijection f defined in eqn. 18 applies in the form:

$$f((\theta_1, 0)(\theta_2, 0), (\theta_3, 0)) = (a, b, c, d) \quad (22)$$

Proof:

According to the definition of function f , we have that

$$\begin{aligned} \cos(\theta_1/2) &= \sqrt{a^2 + b^2} \\ \cos(\theta_2/2) &= a/\cos(\theta_1/2) \\ \cos(\theta_3 + \theta_2/2) &= c/\cos(\theta_1/2) \end{aligned} \quad (23)$$

Example 3 Let us suppose a quantum state, with two qubits initialized, both with $|0\rangle$ and then, with a Hadamard superposition. Thus, the two first components of the vector are equal ($a = b = 1/2$). And let us suppose with an arbitrary number in amplitude c , say $1/3$ as illustrated in fig. 6. The last amplitude is such that the sum of the squares is 1, then the system example is $(1/2, 1/2, 1/3, 1 - \sqrt{1/2 + 1/9})$. Then, the normal form by Bloch spheres is:

$$\begin{aligned} \theta_1 &= 2 \arccos \sqrt{1/4 + 1/4} = \pi/2 \\ \theta_2 &= 2 \arccos (1/2 / \cos(\pi/4)) = \pi/2 \\ \theta_3 + \theta_2/2 &= \arccos (1/3 / \cos(\pi/4)) \\ \theta_3 &= \arccos(\sqrt{2}/6) - \pi/4 \end{aligned} \quad (24)$$

$$|?\rangle = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/3 \\ 1 - \sqrt{1/2 + 1/9} \end{bmatrix}$$

Figure 6: Example of calculating entanglement sphere

$$\begin{aligned} S_1 &= (\pi/2, 0) \\ S_2 &= (\pi/2, 0) \\ S_3 &= (\arccos(\sqrt{2}/6) - \pi/4, 0) \end{aligned} \quad (25)$$

Note that the procedure shown in this example is valid for any value of real amplitude c in a quantum state, according to previous corollary 1. $\square$

We can find an asymmetry in the angles θ of the Bloch spheres, because spheres representing single qubits range, in fact, in the interval $[0, \pi/2]$, because they are divided into 2, but the entanglement sphere is not divided. In the next example, we show the necessity of this

Example 4 Let us suppose a quantum state from a Hadamard superposition, that is $a = b = 1/2$, and values $c = \sqrt{2}/2$, $d = 0$ for the other components. Then, the values of the spheres for representing this system are

$$\begin{aligned} S_1 &= (\pi/2, 0) \\ S_2 &= (\pi/2, 0) \\ S_3 &= (3\pi/4, 0) \end{aligned}$$

Where $\theta_3 > \pi/2$, and we need this, because it is necessary in order to reach value 0 in the amplitude of basis vector $|11\rangle$, obtained with $\sin(\theta_2/2 + \theta_3)$ $\square$

In equation 11 we have introduced the representation model of a 2-qubit system by using 3 Bloch spheres, each one of them described by two real parameters. Then, with theorem 1, we have proved the equivalence between this model and vectorial representation in a 4-dimensional Hilbert space.

This bijective correspondence provides us with the appropriate framework to define a unique Normal Form that describes a two-qubit quantum system using six real coordinates. However, we will introduce a difference for mathematical purposes in order to simplify the notation. In the formula introduced in 11, we first performed a tensor product between the two separate qubits before introducing entanglement, and then applied the controlled rotations. This has a drawback from the perspective of performing calculations, as there may be a phase factor in the initial two qubits that, when the

tensor product is applied, distributes among the components of the vector, complicating subsequent calculations in the controlled rotations.

Nevertheless, as we have already verified in the proof of Theorem 1, the task becomes easier if we separate the calculations of the phases from the calculations of the real parts (moduli). To this end, in the normal form we are about to define, we will first perform the rotations around the Y axis, and then the rotations around the Z axis, both direct and controlled. In this way, we can describe any two-qubit quantum state using a unique normal form, which consists of six rotation gates, four direct and two controlled, applied to a quantum state initialized to the vector $|00\rangle$.

This is described by the following corollary:

Corollary 2 Given a general quantum system of two qubits, there exists a unique normal form to represent it by means of three Bloch spheres

Proof: From eqn.11 and bijection function in eqn. 18

$$\begin{aligned} |Q_1 Q_2\rangle &= \Lambda_2^1 R_z(\varphi_3) \cdot \Lambda_1^0 R_z(\varphi_1) \cdot \Lambda_2^0 R_z(\varphi_2) \cdot \\ &\quad \Lambda_2^1 R_y(\theta_3) \cdot \Lambda_1^0 R_y(\theta_1/2) \cdot \Lambda_2^0 R_y(\theta_2/2) \cdot \\ &\quad |00\rangle \\ &= \cos \theta_1/2 \cdot \cos \theta_2/2 |00\rangle + \\ &\quad \cos \theta_1/2 \cdot \sin \theta_2/2 \cdot e^{i\varphi_2} |01\rangle + \\ &\quad \sin \theta_1/2 \cdot \cos(\theta_2/2 + \theta_3) \cdot e^{i\varphi_1} |10\rangle + \\ &\quad \sin \theta_1/2 \cdot \sin(\theta_2/2 + \theta_3) \cdot e^{i(\varphi_1 + \varphi_2 + \varphi_3)} \cdot \\ &\quad |11\rangle \end{aligned} \quad (26)$$

□

4 CONCLUSION AND FUTURE WORK

In this paper, we have presented a graphical model to describe a two-qubit system using three Bloch spheres: two spheres to model each of the qubits and another to represent entanglement.

We have demonstrated the one-to-one correspondence between the representation using elements of a 4-dimensional Hilbert Space with complex coefficients, where the sum of the squares of the amplitudes is 1, and the representation using three spheres, each one described by two angles (latitude and longitude). In both cases, the representation is achieved using 6 real magnitudes.

This modeling is the starting point of a broader work in which it will be shown that this model can be extended, without any issues, to the general case of n qubits. The work presented in [Wha16], as mentioned earlier, is closely related to this presentation. However,

their modeling using angles is done in an ad hoc manner, without a clear approach that allows for generalization to an arbitrary number of qubits, whereas our model is designed with this generalization in mind.

The generalization has a dual purpose for the future. On the one hand, from a computer science perspective, it will provide an alternative model to the Hilbert Space vector for representing data in a quantum computer simulator, which could offer computational advantages. On the other hand, it will enable the development of a theoretical framework to represent, not only the elements of the Hilbert Space, but also quantum gates themselves, represented by elements of Lie groups of powers of 2 orders, i.e., $SU(2)$, $SU(4)$, $SU(8)$, etc. In this framework, gates are transformations between spheres, i.e., rotations, which will provide a new and interesting framework for studying these models.

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